36 Superspace Approaches to $\mathcal{N} = 1$ Supergravity

$\mathcal{N} = 1$ 超引力的 36 种超空间方法

Sergei M. Kuzenko, Emmanouil S. N. Raptakis, and Gabriele Tartaglino-Mazzucchelli

谢尔盖·M·库津科，埃马努伊尔·S·N·拉普塔基斯，加布里埃莱·塔尔蒂尼诺-马祖凯利

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S. M. Kuzenko (☑) - E. S. N. Raptakis

S. M. 库津科 (☑)- E. S. N. 拉普塔基斯

Department of Physics M013, The University of Western Australia, Perth, WA, Australia e-mail: sergei.kuzenko@uwa.edu. emmanouil.raptakis@research.uwa.edu.au

西澳大学物理系 M013, 澳大利亚西澳大利亚州珀斯邮箱:sergei.kuzenko@uwa.edu.au; emmanouil.raptakis@research.uwa.edu.au

G. Tartaglino-Mazzucchelli

G. 塔尔迪利诺-马祖凯利

School of Mathematics and Physics, University of Queensland, Brisbane, QLD, Australia e-mail: g.tartaglino-mazzucchelli@uq.edu.au

昆士兰大学数学与物理学院, 澳大利亚昆士兰州布里斯班邮箱:g.tartaglino-mazzucchelli@uq.edu.au

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Abstract

摘要

The superspace formalism for $\mathcal{N} = 1$ supergravity in four dimensions is a powerful geometric setting to engineer off-shell supergravity-matter theories, including higher-derivative couplings. This review provides a unified description of the three superspace approaches to $\mathcal{N} = 1$ conformal supergravity: (i) conformal superspace, (ii) U (1) superspace, and (iii) the Grimm-Wess-Zumino formalism. The prepotential formulation for the latter is discussed. We briefly describe the known off-shell formulations for Poincaré and anti-de Sitter supergravity theories as conformal supergravity coupled to certain compensators. As simple applications of the formalism, we present the superfield equations of motion for various off-shell formulations for pure Poincaré and anti-de Sitter supergravity and show that every solution of these equations is also a solution of the equations of motion for conformal supergravity.

四维空间中 $\mathcal{N}=1$ 超引力的超空间形式论是构建脱壳超引力-物质理论 (包括高阶导数耦合) 的强大几何框架。本综述统一描述了 $\mathcal{N}=1$ 共形超引力的三种超空间方案:(i) 共形超空间，(ii) U (1) 超空间，以及 (iii) 格里姆-韦斯-朱米诺形式论。本文还讨论了后者的预备势表述。我们简要介绍了已知的庞加莱和反德西特超引力理论的脱壳表述，即耦合特定补偿子的共形超引力。作为该形式论的简单应用，我们给出了纯庞加莱和纯反德西特超引力不同脱壳表述的超场运动方程，并证明这些方程的每一个解也都是共形超引力运动方程的解。

Keywords

关键词

Superconformal symmetry - Supergravity - Superspace

超共形对称性 - 超引力 - 超空间

Dedicated to the creators of superfield supergravity

献给超场超引力的创立者们

Introduction

引言

Soon after the discovery of $\mathcal{N}=1$ supergravity in four spacetime dimensions [1, 2] (and subsequent construction of the supersymmetric cosmological term [3]), several off-shell formulations for this theory, with different sets of auxiliary fields, were developed. These include the non-minimal [4-6], old minimal [7-10], and new minimal [11, 12] supergravity theories. The most general matter couplings are offered by the old minimal formulation [13, 14]. All interactions constructed in the framework of the new minimal as well as the non-minimal formulations are particular cases of those that can be realised within the old minimal theory [14].

四维时空 [1, 2] 中 $\mathcal{N}=1$ 超引力被发现后 (随后又构造出了超对称宇宙学项 [3])，很快人们就为该理论发展出了多种带有不同辅助场集合的离壳形式，其中包括非极小 [4-6]、旧极小 [7-10] 和新极小 [11,12] 超引力理论。旧极小形式 [13,14] 可以给出最一般的物质耦合。新极小和非极小形式框架下构造出的所有相互作用，都只是旧极小理论中可实现相互作用的特殊情况 [14]。

Traditionally, Refs. [9, 10] are credited with the discovery of old minimal supergravity (see, e.g., [15]), since they have played a fundamental role in the development of supergravity. In fact, this off-shell theory was constructed for the first time using superfield techniques in an unpublished 1977 work by Siegel [7] (which was difficult to digest at the time) and then re-discovered by Wess and Zumino [8], shortly before the publication of [9, 10]. It was explicitly shown [6, 16, 17] that the component reduction of the superfield action for supergravity proposed in [8] coincides with the component actions given in [9, 10]. It was also explicitly demonstrated [6, 18] that the Wess-Zumino action [8] is equivalent to the one proposed in [7]. These are just simple examples of the power of superspace approaches to supergravity.

> 传统上，旧极小超引力的发现归功于文献 [9,10](参见例如 [15])，因为它们对超引力的发展起到了基础性作用。事实上，这种离壳理论最早是西格尔在 1977 年一篇未发表的工作中利用超场技术构造出来的 [7](该工作在当时很难被理解)，之后在 [9,10] 发表前不久，被韦斯和祖米诺重新发现 [8]。已有明确证明 [6, 16, 17]，[8] 中提出的超引力超场作用量的分量约化，与 [9,10] 给出的分量作用量一致。也已有研究明确证明 [6,18]，韦斯-祖米诺作用量 [8] 等价于 [7] 中提出的作用量。这些都只是超空间方法应用于超引力领域威力的简单例证。

Einstein's theory of gravity can be described using a Weyl invariant extension of the Einstein-Hilbert action by means of a compensating scalar field [19, 20]. In other words, ordinary gravity can be thought of as conformal gravity coupled to a compensator. As a generalisation of this idea, Poincaré supergravity can be realised as a locally superconformal invariant theory of supergravity [21], in which the Weyl multiplet of conformal supergravity [22, 23] is coupled to a compensating scalar multiplet [13, 14] (A similar idea was put forward earlier in [7].). It turns out that all known off-shell formulations for Poincaré and anti-de Sitter (AdS) supergravity theories can be recast as conformal supergravity coupled to certain compensating multiplets. Different choices of a compensator correspond to different off-shell formulations for supergravity. This superconformal setting has been developed in the conventional component approach [24-27] under the name "superconformal tensor calculus" and has proved to be truly useful in order to formulate general two-derivative supergravity-matter systems and to study their dynamical properties (see [28] for a review). In our opinion, it becomes especially powerful within superspace formulations for supergravity, which (i) provide remarkably compact expressions for general supergravity-matter actions, (ii) make manifest the geometric properties of such theories, and, most importantly, (iii) offer unique tools to generate higher-derivative couplings in matter-coupled supergravity.

> 爱因斯坦引力理论可以借助补偿标量场，通过爱因斯坦-希尔伯特作用量的外尔不变推广来描述 [19,20]。换句话说，普通引力可以看作耦合了补偿子的共形引力。将这一思想推广，庞加莱超引力可以实现为局域超共形不变的超引力理论 [21]，其中共形超引力的外尔多重态 [22,23] 与补偿标量多重态耦合 [13,14](类似的思想更早被提出于 [7])。研究发现，所有已知的庞加莱和反德西特 (AdS) 超引力离壳形式，都可以重新表述为耦合了特定补偿多重态的共形超引力。补偿子的不同选择对应超引力的不同离壳形式。这种超共形框架在传统分量方法中以“超共形张量演算”的名称被发展起来，已被证明对构造一般双导数超引力-物质系统、研究其动力学性质十分有用 (综述参见 [28])。我们认为，它在超引力的超空间表述中尤为强大:(i) 能为一般超引力-物质作用量给出非常简洁的表达式，(ii) 可以清晰展现这类理论的几何性质，最重要的是，(iii) 能提供独特工具来生成物质耦合超引力中的高阶导数耦合。

There are three fully fledged approaches to describe $\mathcal{N} = 1$ conformal super-gravity in superspace: (i) the Grimm-Wess-Zumino (GWZ) formalism [29] extending the Wess-Zumino formulation for on-shell supergravity [30], (ii) the so-called U(1) superspace proposed by Howe [31], and (iii) the $\mathcal{N} = 1$ conformal superspace approach developed by Butter [32]. Conformal superspace is an ultimate formulation for conformal supergravity in the sense that any different off-shell formulation is either equivalent to it or is obtained from it by partially fixing the gauge freedom. In particular, U (1) superspace can be obtained from a partial gauge fixing of the gauge group in conformal superspace. The $\mathcal{N} = 1$ superconformal tensor calculus reviewed in [28, 33] is also a gauged fixed version of conformal superspace as demonstrated in [32]. Recently, new super-twistor formulations were discovered for conformal supergravity theories in diverse dimensions [34]. In the four-dimensional $\mathcal{N}=1$ case, the supertwistor formulation is expected to be related to conformal superspace; however, relevant technical details have not yet been worked out in the literature.

在超空间中描述 $\mathcal{N}=1$ 共形超引力共有三种成熟的方法:(i) 格林-韦斯-祖米诺 (GWZ) 形式体系 [29]，它推广了壳上超引力的韦斯-祖米诺表述 [30]，(ii) 豪提出的所谓 U(1) 超空间 [31]，(iii) 巴特尔发展的 $\mathcal{N}=1$ 共形超空间方法 [32]。共形超空间是共形超引力的终极表述，任何不同的离壳表述要么等价于它，要么通过部分固定规范自由度从它得到。特别地，U (1) 超空间可以通过固定共形超空间中规范群的部分规范得到。正如文献 [32] 所示，[28, 33] 中综述的 $\mathcal{N}=1$ 超共形张量演算也是共形超空间固定规范后的版本。近年来，人们在不同维度的共形超引力理论中发现了新的超扭量表述 [34]。在四维 $\mathcal{N}=1$ 情形下，超扭量表述预期和共形超空间相关；但文献中尚未梳理出相关的技术细节。

Due to space limitations, in this review, we are not able to discuss many important aspects of $\mathcal{N}=1$ supergravity and its matter couplings. We apologise for the unavoidable omissions and missing references.

受篇幅限制，本综述无法讨论 $\mathcal{N}=1$ 超引力及其物质耦合的许多重要方面，我们对这些不可避免的遗漏和缺引文献深表歉意。

Our two-component spinor notation and conventions follow [35] and are similar to those adopted in [17]. The only difference is that the spinor Lorentz generators $(\sigma_{ab})_\alpha{}^\beta$ and $(\tilde{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta}$ used in [35] have an extra minus sign as compared with [17], specifically $\sigma_{ab}=-\frac{1}{4}(\sigma_a\tilde{\sigma}_b-\sigma_b\tilde{\sigma}_a)$ and $\tilde{\sigma}_{ab}=-\frac{1}{4}(\tilde{\sigma}_a\sigma_b-\tilde{\sigma}_b\sigma_a)$.

我们的双分量旋量记号与约定遵循文献 [35]，与文献 [17] 采用的约定类似。唯一区别在于，文献 [35] 中使用的旋量洛伦兹生成元 $(\sigma_{ab})_\alpha{}^\beta$ 和 $(\tilde{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta}$ 相比文献 [17] 多一个负号，具体为 $\sigma_{ab}=-\frac{1}{4}(\sigma_a\tilde{\sigma}_b-\sigma_b\tilde{\sigma}_a)$ 和 $\tilde{\sigma}_{ab}=-\frac{1}{4}(\tilde{\sigma}_a\sigma_b-\tilde{\sigma}_b\sigma_a)$。

Rigid and Local Superconformal Transformations

刚性与局部超共形变换

In this section we first review the structure of rigid superconformal transformations in Minkowski superspace $\mathbb{M}^{4|4}$. Then, we introduce local superconformal transformations and describe the multiplet of conformal supergravity, following the approach due to Ogievetsky and Sokatchev [36]. It should be pointed out that the superconformal transformations in $\mathbb{M}^{4|4}$ were first studied by Sohnius [37]. Our presentation follows [35].

本节我们首先回顾闵氏超空间中刚性超共形变换的结构 $\mathbb{M}^{4|4}$ 。随后，我们遵循奥格耶夫斯基和索卡切夫提出的方法 [36]，引入局部超共形变换并描述共形超引力多重态。需要指出，$\mathbb{M}^{4|4}$ 中的超共形变换最早由索纽斯研究 [37]。我们的表述参考了文献 [35]。

Rigid Superconformal Transformations

刚性超共形变换

We denote by $z^A = \left(x^a, \theta^\alpha, \bar\theta_{\dot\alpha}\right)$ the Cartesian coordinates for Minkowski super-space $\mathbb{M}^{4|4}$, and use the notation $D_A = (\partial_a, D_\alpha, \bar D^{\dot\alpha})$ for the superspace covariant derivatives. The only nontrivial graded commutation relation is

我们将闵可夫斯基超空间 $\mathbb{M}^{4|4}$ 的笛卡尔坐标记作 $z^A = \left(x^a, \theta^\alpha, \bar\theta_{\dot\alpha}\right)$，并将超空间协变导数记作 $D_A = (\partial_a, D_\alpha, \bar D^{\dot\alpha})$。唯一非平凡的阶化对易关系为

$$\{D_\alpha, \bar D_{\dot\alpha}\} = -2\mathrm{i}(\sigma^b)_{\alpha\dot\alpha}\partial_b = -2\mathrm{i}\partial_{\alpha\dot\alpha}. \tag{1}$$

An infinitesimal superconformal transformation $z^A \to z^A + \delta z^A$, with $\delta z^A = \xi z^A = \left(\xi^a + \mathrm{i}\left(\xi\sigma^a\bar\theta - \theta\sigma^a\bar\xi\right), \xi^\alpha, \bar\xi_{\dot\alpha}\right)$, is generated by a conformal Killing supervector field

满足 $\delta z^A = \xi z^A = \left(\xi^a + \mathrm{i}\left(\xi\sigma^a\bar\theta - \theta\sigma^a\bar\xi\right), \xi^\alpha, \bar\xi_{\dot\alpha}\right)$ 的无穷小超共形变换 $z^A \to z^A + \delta z^A$ 由共形 Killing 超向量场生成

$$\xi = \bar\xi = \xi^b\partial_b + \xi^\beta D_\beta + \bar\xi_{\dot\beta}\bar D^{\dot\beta}. \tag{2}$$

The defining property of ξ is that it takes every chiral superfield Φ to a chiral one,

ξ 的定义性质是它将任意手征超场 Φ 映射为手征超场,

$$\bar D^{\dot\alpha}\Phi = 0 \;\to\; \bar D^{\dot\alpha}(\xi\Phi) = 0. \tag{3}$$

This condition implies the relations

该条件导出如下关系

$$\bar D^{\dot\alpha}\xi^\beta = 0,\; \bar D^{\dot\alpha}\xi^{\dot\beta\beta} = 4\mathrm{i}\varepsilon^{\dot\alpha\dot\beta}\xi^\beta \;\Rightarrow\; \xi^\alpha = -\frac{\mathrm{i}}{8}\bar D_{\dot\alpha}\xi^{\dot\alpha\alpha} \tag{4}$$

and their complex conjugates and therefore

以及它们的复共轭，因此有

$$\bar D_{(\alpha}\xi_{\beta)\dot\beta} = 0,\; \bar D_{(\dot\alpha}\xi_{\beta\dot\beta)} = 0 \;\Rightarrow\; \partial_{(\alpha}\left(\dot\alpha\xi_{\beta)\dot\beta}\right) = 0. \tag{5}$$

It follows that

由此可得

$$[\xi, D_\alpha] = -\left(D_\alpha\xi^\beta\right)D_\beta = -K_\alpha{}^\beta[\xi]D_\beta - \left(\bar\sigma[\xi] - \frac{1}{2}\sigma[\xi]\right)D_\alpha. \tag{6}$$

Here we have introduced chiral Lorentz $\left(K_{\beta\gamma}[\xi] = K_{\gamma\beta}[\xi]\right)$ and super-Weyl $(\sigma[\xi])$ parameters defined by

这里我们引入了由下式定义的手征洛伦兹参数 ($K_{\beta\gamma}[\xi] = K_{\gamma\beta}[\xi]$) 和超外尔参数 ($\sigma[\xi]$)

$$K_{\alpha\beta}[\xi] = D_{(\alpha}\xi_{\beta)}, \ \bar{D}_{\dot{\gamma}}K_{\alpha\beta}[\xi] = 0, \tag{7a}$$

$$\sigma[\xi] = \frac{1}{3}\left(D_\alpha\xi^\alpha + 2\bar{D}^{\dot{\alpha}}\bar{\xi}_{\dot{\alpha}}\right), \ \bar{D}_{\dot{\gamma}}\sigma[\xi] = 0. \tag{7b}$$

We recall that the Lorentz parameters with vector and spinor indices are related to each other as follows: $K^{bc}[\xi] = \left(\sigma^{bc}\right)_{\beta\gamma}K^{\beta\gamma}[\xi] - \left(\tilde{\sigma}^{bc}\right)_{\dot{\beta}\dot{\gamma}}\bar{K}^{\dot{\beta}\dot{\gamma}}[\xi]$.

我们回顾一下，带矢量指标和旋量指标的洛伦兹参数满足如下关系：$K^{bc}[\xi] = \left(\sigma^{bc}\right)_{\beta\gamma}K^{\beta\gamma}[\xi] - \left(\tilde{\sigma}^{bc}\right)_{\dot{\beta}\dot{\gamma}}\bar{K}^{\dot{\beta}\dot{\gamma}}[\xi]$。

The most general conformal Killing supervector field has the form

最一般的共形 Killing 超向量场具有如下形式

$$\xi_+^{\dot{\alpha}\alpha} = a^{\dot{\alpha}\alpha} + \frac{1}{2}(\sigma + \bar{\sigma})y^{\dot{\alpha}\alpha} + \bar{K}^{\dot{\alpha}}{}_{\dot{\beta}}y^{\dot{\beta}\alpha} + y^{\dot{\alpha}\beta}K_\beta{}^\alpha - y^{\dot{\alpha}\beta}b_{\beta\dot{\beta}}y^{\dot{\beta}\alpha}$$

$$+4\mathrm{i}\bar{\varepsilon}^{\dot{\alpha}}\theta^\alpha - 4y^{\dot{\alpha}\beta}\eta_\beta\theta^\alpha \tag{8a}$$

$$\xi^\alpha = \varepsilon^\alpha + \left(\bar{\sigma} - \frac{1}{2}\sigma\right)\theta^\alpha + \theta^\beta K_\beta{}^\alpha - \theta^\beta b_{\beta\dot{\beta}}y^{\dot{\beta}\alpha} - \mathrm{i}\bar{\eta}_{\dot{\beta}}y^{\dot{\beta}\alpha} + 2\theta^2\eta^\alpha, \tag{8b}$$

where we have introduced the complex four-vector

其中我们引入了复四矢量

$$\xi_+^a = \xi^a + 2\mathrm{i}\xi\sigma^a\bar{\theta}, \ \bar{\xi}^a = \xi^a, \tag{9}$$

along with the complex bosonic coordinates $y^a = x^a + \mathrm{i}\theta\sigma^a\bar{\theta}$ of the chiral subspace of $\mathbb{M}^{4|4}$. The constant bosonic parameters in (8) correspond to the spacetime translation $\left(a^{\dot{\alpha}\alpha}\right)$, Lorentz transformation $\left(K_\beta{}^\alpha, \bar{K}^{\dot{\alpha}}{}_{\dot{\beta}}\right)$, special conformal transformation $\left(b_{\alpha\dot{\beta}}\right)$, and combined scale and R-symmetry transformations $\left(\sigma = \tau - \frac{2}{3}\mathrm{i}\varphi\right)$. The constant fermionic parameters in (8) correspond to the Q-supersymmetry (ε^α) and S-supersymmetry (η_α) transformations. The constant parameters $K_{\alpha\beta}$ and σ are obtained from $K_{\alpha\beta}[\xi]$ and $\sigma[\xi]$, respectively, by setting $z^A = 0$.

以及 $\mathbb{M}^{4|4}$ 的手征子空间的复玻色坐标 $y^a = x^a + \mathrm{i}\theta\sigma^a\bar{\theta}$。式 (8) 中的常数玻色参数分别对应时空平移 $\left(a^{\dot{\alpha}\alpha}\right)$、洛伦兹变换 $\left(K_\beta{}^\alpha, \bar{K}^{\dot{\alpha}}{}_{\dot{\beta}}\right)$、特殊共形变换 $\left(b_{\alpha\dot{\beta}}\right)$，以及标度与 R 对称的联合变换 $\left(\sigma = \tau - \frac{2}{3}\mathrm{i}\varphi\right)$。式 (8) 中的常数费米参数分别对应 Q 超对称性变换 (ε^α) 和 S 超对称性变换 (η_α)。常数参数 $K_{\alpha\beta}$ 和 σ 可分别由 $K_{\alpha\beta}[\xi]$ 和 $\sigma[\xi]$ 令 $z^A = 0$ 得到。

It is convenient to introduce a condensed notation for the superconformal parameters

为超共形参数引入简写记号是很方便的

$$\lambda^{\tilde{a}} = \left(a^A, K^{ab}, \tau, \varphi, b_A\right),\ a^A := \left(a^a, \varepsilon^\alpha, \bar{\varepsilon}_{\dot{\alpha}}\right),\ b_A := \left(b_a, \eta_\alpha, \overline{\eta}^{\dot{\alpha}}\right), \tag{10}$$

as well as for the generators of the superconformal group

对超共形群的生成元也可引入类似简写

$$X_{\tilde{a}} = \left(P_A, M_{ab}, \mathbb{D}, \mathbb{Y}, K^A\right),\ P_A := \left(P_a, Q_\alpha, \bar{Q}^{\dot{\alpha}}\right),\ K^A := \left(K^a, S^\alpha, \bar{S}_{\dot{\alpha}}\right).$$

(11)

The general conformal Killing supervector field on $\mathbb{C}^{4|2}$,

$\mathbb{C}^{4|2}$ 上的一般共形 Killing 超向量场,

$$\xi = \xi_+^a(y,\theta)\frac{\partial}{\partial y^a} + \xi^\alpha(y,\theta)\frac{\partial}{\partial\theta^\alpha} \equiv \xi_+^a\partial/\partial y^a + \xi^\alpha\partial_\alpha, \tag{12}$$

may be written in the form:

可以写成如下形式:

$$\xi = \lambda^{\tilde{a}}\xi_{\tilde{a}}(X) = a^A\xi_A(P) + \frac{1}{2}K^{ab}\xi_{ab}(M) + \tau\xi(\mathbb{D}) + \mathrm{i}\varphi\xi(\mathbb{Y}) + b_A\xi^A(K). \tag{13}$$

We read off the relevant supervector fields:

我们可以直接读出相关的超向量场:

$$\xi_a(P) = \partial/\partial y^a,\ \xi_\alpha(P) = \partial_\alpha,\ \overline{\xi}^{\dot{\alpha}}(P) = -2\mathrm{i}(\tilde{\sigma}^c\theta)^{\dot{\alpha}}\partial/\partial y^c, \tag{14a}$$

$$\xi_{ab}(M) = y_a\partial/\partial y^b - y_b\partial/\partial y^a + (\theta\sigma_{ab})^\gamma\partial_\gamma, \tag{14b}$$

$$\xi(\mathbb{D}) = y^c\partial/\partial y^c + \frac{1}{2}\theta^\gamma\partial_\gamma,\ \xi(\mathbb{Y}) = \theta^\gamma\partial_\gamma, \tag{14c}$$

$$\xi^a(K) = 2y^ay^c\partial/\partial y^c - y^2\partial/\partial y_a - (\theta\sigma^a\tilde{\sigma}^c)^\gamma y_c\partial_\gamma, \tag{14d}$$

$$\xi^\alpha(K) = 2\left(\theta\sigma^c\tilde{\sigma}^d\right)^\alpha y_d\partial/\partial y^c - 2\theta^2\varepsilon^{\alpha\gamma}\partial_\gamma, \tag{14e}$$

$$\overline{\xi}_{\dot{\alpha}}(K) = \mathrm{i}(\sigma^c)^\gamma{}_{\dot{\alpha}}y_c\partial_\gamma. \tag{14f}$$

Making use of the above operators, we derive the graded commutation relations for the superconformal algebra, $\left[X_{\tilde{a}}, X_{\tilde{b}}\right\} = -f_{\tilde{a}\tilde{b}}{}^{\tilde{c}}X_{\tilde{c}}$, keeping in mind the relation

利用上述算符，我们推导超共形代数的分次对易关系，$\left[X_{\tilde{a}}, X_{\tilde{b}}\right\} = -f_{\tilde{a}\tilde{b}}{}^{\tilde{c}} X_{\tilde{c}}$ ，牢记该关系

$$\xi = \lambda^{\tilde{\alpha}} \xi_{\tilde{a}}(X) \to \delta_{\xi} = \lambda^{\tilde{a}} X_{\tilde{a}}, \ [\xi_1, \xi_2] \to -\left[\delta_{\xi_1}, \delta_{\xi_2}\right]. \tag{15}$$

We start with the commutation relations for the conformal algebra:

我们从共形代数的对易关系开始:

$$[M_{ab}, M_{cd}] = 2\eta_{c[a} M_{b]d} - 2\eta_{d[a} M_{b]c}, \tag{16a}$$

$$[M_{ab}, P_c] = 2\eta_{c[a} P_{b]}, \ [\mathbb{D}, P_a] = P_a, \tag{16b}$$

$$[M_{ab}, K_c] = 2\eta_{c[a} K_{b]}, \ [\mathbb{D}, K_a] = -K_a, \tag{16c}$$

$$[K_a, P_b] = 2\eta_{ab}\mathbb{D} + 2M_{ab}. \tag{16d}$$

The R -symmetry generator $\mathbb{Y}$ commutes with all the generators of the conformal group. In four dimensions, the superconformal algebra is obtained by extending the translation generator to P_A and the special conformal generator to K^A . The commutation relations involving the Q -supersymmetry generators with the bosonic ones are:

R 对称的生成元 $\mathbb{Y}$ 与共形群的所有生成元对易。在四维空间中，超共形代数是通过将平移生成元推广为 P_A 、将特殊共形生成元推广为 K^A 得到的。涉及 Q 超对称生成元与玻色生成元的对易关系为:

$$\left[M_{ab}, Q_\gamma\right] = (\sigma_{ab})_\gamma{}^\delta Q_\delta, \ \left[M_{ab}, \bar{Q}^{\dot{\gamma}}\right] = (\tilde{\sigma}_{ab})^{\dot{\gamma}}{}_{\dot{\delta}} \bar{Q}^{\dot{\delta}}, \tag{17a}$$

$$[\mathbb{D}, Q_\alpha] = \frac{1}{2} Q_\alpha, \ \left[\mathbb{D}, \bar{Q}^{\dot{\alpha}}\right] = \frac{1}{2} \bar{Q}^{\dot{\alpha}}, \tag{17b}$$

$$[\mathbb{Y}, Q_\alpha] = Q_\alpha, \ \left[\mathbb{Y}, \bar{Q}^{\dot{\alpha}}\right] = -\bar{Q}^{\dot{\alpha}}, \tag{17c}$$

$$\left[K^a, Q_\beta\right] = -\mathrm{i}(\sigma^a)_\beta{}^{\dot{\beta}} \bar{S}_{\dot{\beta}}, \ \left[K^a, \bar{Q}^{\dot{\beta}}\right] = -\mathrm{i}(\sigma^a)^{\dot{\beta}}{}_\beta S^\beta. \tag{17d}$$

The commutation relations involving the S -supersymmetry generators with the bosonic operators are:

涉及 S 超对称生成元与玻色算符的对易关系为:

$$[M_{ab}, S^\gamma] = -(\sigma_{ab})_\beta{}^\gamma S^\beta, \ \left[M_{ab}, \bar{S}_{\dot{\gamma}}\right] = -(\tilde{\sigma}_{ab})^{\dot{\beta}}{}_{\dot{\gamma}} \bar{S}_{\dot{\beta}}, \tag{18a}$$

$$[\mathbb{D}, S^\alpha] = -\frac{1}{2} S^\alpha, \ \left[\mathbb{D}, \bar{S}_{\dot{\alpha}}\right] = -\frac{1}{2} \bar{S}_{\dot{\alpha}}, \tag{18b}$$

$$[\mathbb{Y}, S^\alpha] = -S^\alpha, \ \left[\mathbb{Y}, \bar{S}_{\dot{\alpha}}\right] = \bar{S}_{\dot{\alpha}}, \tag{18c}$$

$$[S^{\alpha}, P_b] = \mathrm{i}(\sigma_b)^{\alpha}{}_{\dot{\beta}}\bar{Q}^{\dot{\beta}},\ [\bar{S}_{\dot{\alpha}}, P_b] = \mathrm{i}(\sigma_b)_{\dot{\alpha}}{}^{\beta}Q_{\beta}. \tag{18d}$$

Finally, the anti-commutation relations of the fermionic generators are:

最后，费米子生成元的反对易关系为:

$$\{Q_{\alpha}, \bar{Q}^{\dot{\alpha}}\} = -2\mathrm{i}(\sigma^b)_{\alpha}{}^{\dot{\alpha}}P_b = -2\mathrm{i}P_{\alpha}{}^{\dot{\alpha}}, \tag{19a}$$

$$\{S^{\alpha}, \bar{S}_{\dot{\alpha}}\} = 2\mathrm{i}(\sigma^b)^{\alpha}\dot{\alpha}K_b = 2\mathrm{i}K^{\alpha}\dot{\alpha}, \tag{19b}$$

$$\{S^{\alpha}, Q_{\beta}\} = 2\delta^{\alpha}_{\beta}\mathbb{D} - 4M^{\alpha}{}_{\beta} - 3\delta^{\alpha}_{\beta}\mathbb{Y}, \tag{19c}$$

$$\{\bar{S}_{\dot{\alpha}}, \bar{Q}^{\dot{\beta}}\} = 2\delta^{\dot{\beta}}_{\dot{\alpha}}\mathbb{D} + 4\bar{M}_{\dot{\alpha}}{}^{\dot{\beta}} + 3\delta^{\dot{\beta}}_{\dot{\alpha}}\mathbb{Y}, \tag{19d}$$

where $M_{\alpha\beta} = \frac{1}{2}(\sigma^{ab})_{\alpha\beta}M_{ab}$ and $\bar{M}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2}(\tilde{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}}M_{ab}$. Note that all remaining (anti-)commutators not explicitly listed above vanish identically.

其中 $M_{\alpha\beta} = \frac{1}{2}(\sigma^{ab})_{\alpha\beta}M_{ab}$ 和 $\bar{M}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2}(\tilde{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}}M_{ab}$ 。注意，所有上文未明确列出的其余 (反) 对易子均恒为零。

The graded commutation relations (16)-(19) constitute the $\mathcal{N} = 1$ superconfor-mal algebra, $\mathfrak{su}(2,2 \mid 1)$. Its generators obey the graded Jacobi identity

分次对易关系 (16)-(19) 构成了 $\mathcal{N} = 1$ 超共形代数，$\mathfrak{su}(2,2 \mid 1)$ 。其生成元满足分次雅可比恒等式

$$(-1)^{\varepsilon_{\tilde{a}}\varepsilon_{\tilde{c}}}\left[X_{\tilde{a}}, [X_{\tilde{b}}, X_{\tilde{c}}\}\right\} + \text{ (two cycles) } = 0, \tag{20}$$

where $\varepsilon_{\tilde{a}} = \varepsilon(X_{\tilde{a}})$ is the Grassmann parity of the generator $X_{\tilde{a}}$. Making use of $[X_{\tilde{a}}, X_{\tilde{b}}\} = -f_{\tilde{a}\tilde{b}}{}^{\tilde{c}}X_{\tilde{c}}$, the Jacobi identities are equivalently written as

其中 $\varepsilon_{\tilde{a}} = \varepsilon(X_{\tilde{a}})$ 是生成元 $X_{\tilde{a}}$ 的格拉斯曼奇偶性。利用 $[X_{\tilde{a}}, X_{\tilde{b}}\} = -f_{\tilde{a}\tilde{b}}{}^{\tilde{c}}X_{\tilde{c}}$ ， 雅可比恒等式可以等价写为

$$f_{[\tilde{a}\tilde{b}}{}^{\tilde{d}}f_{|\tilde{d}|\tilde{c}\}}{}^{\tilde{e}} = 0. \tag{21}$$

It remains to discuss superconformal transformation laws for superfields. Here we restrict our discussion to primary superfields. Given a conformal Killing supervector field ξ , the corresponding infinitesimal superconformal transformation acts on a primary tensor superfield U (with suppressed indices) by the rule

接下来我们讨论超场的超共形变换规律。此处我们仅讨论基本超场: 给定共形基灵超向量场 ξ ， 对应的无穷小超共形变换按以下规则作用在基本张量超场 U (隐去指标) 上

$$\delta_{\xi}U = \mathcal{K}[\xi]U,\ \mathcal{K}[\xi] = \xi + \frac{1}{2}K^{ab}[\xi]M_{ab} + p\sigma[\xi] + q\bar{\sigma}[\xi]. \tag{22}$$

Here the parameters p and q are related to the dimension (Weyl weight) w and $\mathrm{U}(1)_R$ charge c of U as follows: $w = p + q$ and $p - q = -\frac{3}{2}c$. The Lorentz generators M_{ab} in (22) act on the indices of U . The commutation relation for these matrices differs by overall sign from (16a). This is due to the fact that, in conformal (super)gravity, subsequent transformations are applied as follows: $\delta_{\xi_2}\delta_{\xi_1}U = \delta_{\xi_2}\mathcal{K}[\xi_1]U = \mathcal{K}[\xi_1]\delta_{\xi_2}U = \mathcal{K}[\xi_1]\mathcal{K}[\xi_2]U$, see [28] for more details.

此处参数 p 和 q 与 U 的维数 (外尔权重) w 和 $\mathrm{U}(1)_R$ 电荷 c 满足以下关系: $w = p+q$ 和 $p-q = -\frac{3}{2}c$。(22) 中的洛伦兹生成元 M_{ab} 作用在 U 的指标上。这些矩阵的对易关系与 (16a) 仅相差一个整体符号，这是因为在共形 (超) 引力中，后续变换按下述方式作用: $\delta_{\xi_2}\delta_{\xi_1}U = \delta_{\xi_2}\mathcal{K}[\xi_1]U = \mathcal{K}[\xi_1]\delta_{\xi_2}U = \mathcal{K}[\xi_1]\mathcal{K}[\xi_2]U$，更多细节参见文献 [28]。

Superconformal Transformations and Complex Geometry

超共形变换与复几何

Minkowski superspace $\mathbb{M}^{4|4}$ is embedded in the so-called chiral superspace $\mathbb{C}^{4|2}$, parametrised by complex coordinates y^a and θ^α , as the real surface

闵可夫斯基超空间 $\mathbb{M}^{4|4}$ 被嵌入所谓手征超空间 $\mathbb{C}^{4|2}$，由复坐标 y^a 和 θ^α 参数化，作为实曲面

$$\frac{1}{2}(y^a - \bar{y}^a) = \mathrm{i}\theta\sigma^a\bar{\theta}, \ \frac{1}{2}(y^a + \bar{y}^a) = x^a. \tag{23}$$

This is a special member of a family of real superspaces $\mathcal{M}^{4|4}(\mathcal{H})$ embedded in $\mathbb{C}^{4|2}$ by the rule

这是由如下规则嵌入在 $\mathbb{C}^{4|2}$ 中的一族实超空间 $\mathcal{M}^{4|4}(\mathcal{H})$ 的特殊成员

$$\frac{1}{2}(y^a - \bar{y}^a) = \mathrm{i}\mathcal{H}^a\left(x,\theta,\bar{\theta}\right), \ \frac{1}{2}(y^a + \bar{y}^a) = x^a, \tag{24}$$

where the four real bosonic functions $\mathcal{H}^a\left(x,\theta,\bar{\theta}\right)$ may be arbitrary. With respect to the super-Poincaré transformations

其中四个实玻色子函数 $\mathcal{H}^a\left(x,\theta,\bar{\theta}\right)$ 可以任意选取。关于超庞加莱变换

$$\delta y^a = a^a - K^a{}_b y^b + 2\mathrm{i}\theta\sigma^a\bar{\varepsilon}, \ \delta\theta^\alpha = \varepsilon^\alpha + \frac{1}{2}K^{bc}(\theta\sigma_{bc})^\alpha, \tag{25}$$

$\mathcal{H}^a(z) - \theta\sigma^a\bar{\theta}$ proves to be a vector superfield. What is special about Minkowski superspace, $\mathcal{M}^{4|4}\left(\theta\sigma\bar{\theta}\right)$, is the fact that (23) is the unique surface of the type $\mathcal{M}^{4|4}(\mathcal{H})$ which is invariant under the super-Poincaré transformations (25).

$\mathcal{H}^a(z) - \theta\sigma^a\bar{\theta}$ 被证明是一个向量超场。对于闵可夫斯基超空间 $\mathcal{M}^{4|4}\left(\theta\sigma\bar{\theta}\right)$，其特殊之处在于 (23) 是 $\mathcal{M}^{4|4}(\mathcal{H})$ 类型中唯一在超庞加莱变换 (25) 下不变的曲面。

It turns out that the superconformal transformations (8) are the most general holomorphic transformations on $\mathbb{C}^{4|2}$ of the form

可以证明，超共形变换 (8) 是 $\mathbb{C}^{4|2}$ 上如下形式的最一般全纯变换

$$\delta y^{a} = \lambda^{a}(y,\theta)\,,\ \delta\theta^{\alpha} = \lambda^{\alpha}(y,\theta)\,, \tag{26}$$

which leave invariant the superspace $\mathbb{M}^{4|4}$ defined by (23). This remarkable result indicates that (i) arbitrary holomorphic transformations (26) should be interpreted as local superconformal ones and (ii) $\mathcal{H}^{a}\left(x,\theta,\overline{\theta}\right)$ should be used to describe conformal supergravity, a supersymmetric extension of conformal gravity.

它保持由 (23) 定义的超空间 $\mathbb{M}^{4|4}$ 不变。这一显著结论表明:(i) 任意全纯变换 (26) 都应被解释为局部超共形变换；(ii) 应使用 $\mathcal{H}^{a}\left(x,\theta,\overline{\theta}\right)$ 描述共形超引力，即共形引力的超对称推广。

Local Superconformal Transformations

局域超共形变换

Following [36], the gauge group of conformal supergravity is postulated to be the supergroup of holomorphic reparametrisations of $\mathbb{C}^{4|2}$

根据文献 [36]，共形超引力的规范群被假定为 $\mathbb{C}^{4|2}$ 的全纯重参数化超群

$$y^{m} \to y'^{m} = f^{m}(y,\theta)\,,\ \theta^{\mu} \to \theta'^{\mu} = f^{\mu}(y,\theta)\,,\ \mathrm{Ber}\left(\frac{\partial\left(y',\theta'\right)}{\partial(y,\theta)}\right) \neq 0. \tag{27}$$

In curved superspace, we distinguish between curved and flat-space indices. Latin and Greek letters from the middle of each alphabet are used for curved-space indices. Letters from the beginning of each alphabet denote flat-space indices.

在弯曲超空间中，我们区分弯曲指标与平直空间指标。每个字母表中位于中部的拉丁字母和希腊字母用于弯曲空间指标，位于起始位置的字母表示平直空间指标。

In practise, it suffices to work with infinitesimal holomorphic transformations,

实际上，只需处理无穷小全纯变换,

$$y^{m} \to y'^{m} = y^{m} - \lambda^{m}(y,\theta)\,,\ \theta^{\mu} \to \theta'^{\mu} = \theta^{\mu} - \lambda^{\mu}(y,\theta)\,. \tag{28}$$

When restricted to $\mathcal{M}^{4|4}(\mathcal{H})$, this transformation acts as follows:

当限制在 $\mathcal{M}^{4|4}(\mathcal{H})$ 上时，该变换的作用如下:

$$x^{m} \to x'^{m} = x^{m} - \frac{1}{2}\lambda^{m}(x+\mathrm{i}\mathcal{H},\theta) - \frac{1}{2}\overline{\lambda}^{m}\left(x-\mathrm{i}\mathcal{H},\overline{\theta}\right), \tag{29a}$$

$$\theta^{\mu} \to \theta'^{\mu} = \theta^{\mu} - \lambda^{\mu}(x + \mathrm{i}\mathcal{H}, \theta), \tag{29b}$$

as well as

以及

$$\mathcal{H}'^{m}\left(x', \theta', \overline{\theta}'\right) = \mathcal{H}^{m}\left(x, \theta, \overline{\theta}\right) + \frac{\mathrm{i}}{2}\lambda^{m}(x + \mathrm{i}\mathcal{H}, \theta) - \frac{\mathrm{i}}{2}\overline{\lambda}^{m}\left(x - \mathrm{i}\mathcal{H}, \overline{\theta}\right). \tag{29c}$$

From here, it follows that $\delta\mathcal{H}^{m} = \mathcal{H}'^{m}\left(x, \theta, \overline{\theta}\right) - \mathcal{H}^{m}\left(x, \theta, \overline{\theta}\right)$ is given by

由此可得，$\delta\mathcal{H}^{m} = \mathcal{H}'^{m}\left(x, \theta, \overline{\theta}\right) - \mathcal{H}^{m}\left(x, \theta, \overline{\theta}\right)$ 由下式给出

$$\delta\mathcal{H}^{m} = \frac{\mathrm{i}}{2}\left(\lambda^{m} - \overline{\lambda}^{m}\right) + \left(\frac{1}{2}\left(\lambda^{n} + \overline{\lambda}^{n}\right)\partial_{n} + \lambda^{\mu}\partial_{\mu} + \overline{\lambda}_{\dot{\mu}}\partial^{\dot{\mu}}\right)\mathcal{H}^{m},$$

$$\lambda^{m} = \lambda^{m}(x + \mathrm{i}\mathcal{H}, \theta), \ \lambda^{\mu} = \lambda^{\mu}(x + \mathrm{i}\mathcal{H}, \theta). \tag{30}$$

This is the gauge transformation law of $\mathcal{H}^{m}$.

这就是 $\mathcal{H}^{m}$ 的规范变换定律。

Making use of the gauge freedom for $\mathcal{H}^{m}$ allows one to choose a gauge condition

利用 $\mathcal{H}^{m}$ 的规范自由度可以选择规范条件

$$\mathcal{H}^{m}\left(x, \theta, \overline{\theta}\right) = \theta\sigma^{a}\overline{\theta}e_{a}{}^{m}(x) - \mathrm{i}\overline{\theta}^{2}\theta^{\alpha}\Psi_{\alpha}^{m}(x) + \mathrm{i}\theta^{2}\overline{\theta}_{\dot{\alpha}}\overline{\Psi}^{m\dot{\alpha}}(x)$$

$$+\theta^{2}\overline{\theta}^{2}\left(A^{m}(x) - \frac{1}{4}e_{a}{}^{m}(x)\varepsilon^{abcd}\omega_{bcd}(x)\right). \tag{31}$$

Here $\omega_{abc} = -\omega_{acb} = e_{a}{}^{m}\omega_{mbc}$ is the torsion-free Lorentz connection associated with the vielbein $e^{a} = \mathrm{d}x^{m}e_{m}{}^{a}$ and its dual frame field $e_{a} = e_{a}{}^{m}\partial_{m}$:

此处 $\omega_{abc} = -\omega_{acb} = e_{a}{}^{m}\omega_{mbc}$ 是与标架 $e^{a} = \mathrm{d}x^{m}e_{m}{}^{a}$ 及其对偶标架场 $e_{a} = e_{a}{}^{m}\partial_{m}$ 关联的无挠洛伦兹联络:

$$\omega_{abc} = -\frac{1}{2}\left(\mathcal{C}_{bca} + \mathcal{C}_{acb} - \mathcal{C}_{abc}\right), \ [e_{a}, e_{b}] = \mathcal{C}_{ab}{}^{c}e_{c}. \tag{32}$$

Additionally, it will be shown below that A^{m} is the $\mathrm{U}(1)$ gauge field.

此外，下文将证明 A^{m} 是 $\mathrm{U}(1)$ 规范场。

The residual gauge freedom, which preserves the condition (31), is given by

保留条件 (31) 的剩余规范自由度由下式给出

$$\lambda^m(\theta) = \mathfrak{a}^m + 2\mathrm{i}\theta\sigma^a\bar{\varepsilon}e_a{}^m - 2\theta^2\bar{\varepsilon}\overline{\Psi}^m,\ \bar{\mathfrak{a}}^m = \mathfrak{a}^m, \tag{33a}$$

$$\lambda^\alpha(\theta) = \varepsilon^\alpha + \theta^\alpha\left(\frac{1}{2}\tau + \mathrm{i}\varphi\right) + \theta^\beta K_\beta{}^\alpha$$

$$+\theta^2\left[\eta^\alpha - \frac{1}{2}(\bar{\varepsilon}\tilde{\sigma}^a)^\alpha\left(\mathrm{i}\omega^b{}_{ba} + \frac{1}{2}\varepsilon^{abcd}\omega_{bcd}\right)\right], \tag{33b}$$

with $K_\alpha{}^\beta = \frac{1}{2}K^{ab}(\sigma_{ab})_\alpha{}^\beta$. Keeping in mind the structure of conformal Killing supervector fields (8), we can give the following interpretations to the gauge parameters (33). The bosonic parameters correspond to the general coordinate $(\mathfrak{a}^m)$, local Lorentz $(K_\alpha{}^\beta)$, Weyl (τ) , and local chiral (φ) transformations. The fermionic parameters correspond to the local Q -supersymmetry (ε^α) and S -supersymmetry (η^α) transformations.

满足 $K_\alpha{}^\beta = \frac{1}{2}K^{ab}(\sigma_{ab})_\alpha{}^\beta$ 。结合共形 Killing 超向量场 (8) 的结构，我们可以对规范参数 (33) 给出如下解释: 玻色参数对应一般坐标 $(\mathfrak{a}^m)$、局域洛伦兹 $(K_\alpha{}^\beta)$、外尔 (τ) 和局域手征 (φ) 变换；费米参数对应局域 Q 超对称性 (ε^α) 和 S 超对称性 (η^α) 变换。

The superfield gauge transformation (30) allows us to work out the transformation laws of the component fields in (31). Choosing $\mathfrak{a}^m \neq 0$ and switching off the other parameters in (33) gives

超场规范变换 (30) 允许我们推导出 (31) 中分量场的变换定律。选定 $\mathfrak{a}^m \neq 0$ 并令 (33) 中其他参数为零，可得

$$\delta_\mathfrak{a} e_a = [\mathfrak{a}, e_a],\ \delta_\mathfrak{a}\Psi_\alpha = [\mathfrak{a}, \Psi_\alpha],\ \delta_\mathfrak{a} A = [\mathfrak{a}, A], \tag{34}$$

where we have introduced the first-order operators $\mathfrak{a} = \mathfrak{a}^m\partial_m, \Psi_\alpha = \Psi_\alpha^m\partial_m$ and $A = A^m\partial_m$. Next, choosing $K_\alpha{}^\beta \neq 0$ and switching off the other parameters in (33) gives

其中我们引入了一阶算符 $\mathfrak{a} = \mathfrak{a}^m\partial_m, \Psi_\alpha = \Psi_\alpha^m\partial_m$ 和 $A = A^m\partial_m$ 。接下来，选定 $K_\alpha{}^\beta \neq 0$ 并令 (33) 中其他参数为零，可得

$$\delta_K e_a = K_a{}^b e_b,\ \delta_K\Psi_\alpha = K_\alpha{}^\beta\Psi_\beta,\ \delta_K A = 0. \tag{35}$$

The transformation laws (34) and (35) allow us to interpret the field $e_a{}^m$ as the inverse vielbein. They also show that Ψ_α^m transforms as a world vector and a Weyl spinor, while A^m is a vector field. Next, choosing $\frac{1}{2}\tau + \mathrm{i}\varphi \neq 0$ and switching off the other parameters in (33) gives the Weyl (τ) and local chiral (φ) transformations

变换定律 (34) 和 (35) 表明，场 $e_a{}^m$ 是逆标架，同时也说明 Ψ_α^m 按照世界向量和外尔旋量变换，而 A^m 是向量场。接下来，选定 $\frac{1}{2}\tau + \mathrm{i}\varphi \neq 0$ 并令 (33) 中其他参数为零，可得外尔 (τ) 和局域手征 (φ) 变换

$$\delta_\tau e_a{}^m = \tau e_a{}^m,\ \delta_\tau\Psi_\alpha^m = \frac{3}{2}\tau\Psi_\alpha^m,\ \delta_\tau A_m = 0; \tag{36}$$

$$\delta_\varphi e_a{}^m = 0,\ \delta_\varphi\Psi_\alpha^m = -\mathrm{i}\varphi\Psi_\alpha^m,\ \delta_\varphi A_m = \partial_m\varphi. \tag{37}$$

Here we have introduced the one-form $A_m = g_{mn}A^n$, where $g_{mn}(x) = e_m{}^a(x)e_n{}^b(x)\eta_{ab}$ is the Lorentzian metric associated with the vielbein $e_m{}^a$. It follows that A_m is the gauge field for the chiral $\mathrm{U}(1)_R$ group. Since $\mathcal{H}^m$ contains the inverse vielbein at the component level, it is called the gravitational superfield.

此处我们引入了一元型 $A_m = g_{mn}A^n$ ，其中 $g_{mn}(x) = e_m{}^a(x)e_n{}^b(x)\eta_{ab}$ 是与标架 $e_m{}^a$ 关联的洛伦兹度规。由此可知 A_m 是手征 $\mathrm{U}(1)_R$ 群的规范场。由于 $\mathcal{H}^m$ 在分量层面包含逆标架，因此它被称为引力超场。

It remains to consider local supersymmetry transformations. Choosing $\eta^\alpha \neq 0$ and switching off the other parameters in (33) gives the S -supersymmetry transformation laws

接下来我们讨论局域超对称变换。令 $\eta^\alpha \neq$ 为 0 并消去 (33) 式中的其他参数，即可得到 S 超对称变换定律

$$\delta_\eta e_a{}^m = 0,\ \delta_\eta \Psi^m_\alpha = e_a{}^m(\sigma^a\bar{\eta})_\alpha,\ \delta_\eta A^m = \mathrm{i}\left(\bar{\eta}\overline{\Psi}^m - \eta\Psi^m\right). \tag{38}$$

Finally, for the Q -supersymmetry transformation, we obtain

最后，对于 Q 超对称变换，我们得到

$$\delta_\varepsilon e^m_a = \mathrm{i}\left(\Psi^m\sigma_a\bar{\varepsilon} - \varepsilon\sigma_a\overline{\Psi}^m\right), \tag{39a}$$

$$\delta_\varepsilon \Psi^m_\alpha = -\left(\sigma^a\tilde{\sigma}^b\nabla_a\varepsilon\right)_\alpha e_b{}^m + 2\mathrm{i}A^m\varepsilon_\alpha, \tag{39b}$$

$$\delta_\varepsilon A^m = -\frac{\mathrm{i}}{4}e_a{}^m\varepsilon^{abcd}\nabla_b\left(\varepsilon\sigma_c\overline{\Psi}^n - \Psi^n\sigma_c\bar{\varepsilon}\right)e_{nd} - \frac{1}{2}\nabla_a\left(\varepsilon\sigma^a\overline{\Psi}^m + \Psi^m\sigma^a\bar{\varepsilon}\right)$$

$$+\left(\nabla_n\varepsilon\sigma^a\overline{\Psi}^n + \Psi^n\sigma^a\nabla_n\bar{\varepsilon}\right)e_a{}^m. \tag{39c}$$

Here ∇_n and $\nabla_a = e_a{}^n\nabla_n$ are standard torsion-free covariant derivatives, in particular

此处 ∇_n 和 $\nabla_a = e_a{}^n\nabla_n$ 是标准的无挠协变导数，具体来说

$$\nabla_n e_a{}^m = 0,\ \nabla_n\Psi^m_\alpha = \partial_n\Psi^m_\alpha - \frac{1}{2}\omega_n{}^{bc}(\sigma_{bc})_\alpha{}^\beta\Psi^m_\beta + \Gamma^m_{nr}\Psi^r_\alpha, \tag{40}$$

where the Lorentz connection is given by (32) and Γ^m_{nr} denotes the Christoffel symbols.

其中洛伦兹联络由 (32) 式给出，Γ^m_{nr} 表示克里斯托费尔符号。

The S - and Q -supersymmetry transformation laws can be rewritten in a more convenient and familiar form, if the dynamical fields $e_a{}^m, \Psi^m_\alpha$, and A^m are replaced with $e_m{}^a, \Psi_{m\alpha} = g_{mn}\Psi^n_\alpha$ and

如果将动力学场 $e_a{}^m, \Psi^m_\alpha$ 和 A^m 替换为 $e_m{}^a, \Psi_{m\alpha} = g_{mn}\Psi^n_\alpha$ 和，S 与 Q 超对称变换定律可以改写为更方便、更常见的形式

$$\mathfrak{A}_m = A_m - \frac{1}{2}\Psi^n\sigma_m\overline{\Psi}_n + \frac{1}{2}\left(\Psi_m\sigma_n\overline{\Psi}^n + \Psi^n\sigma_n\overline{\Psi}_m\right) + \frac{\mathrm{i}}{8}e_m{}^a\varepsilon_{abcd}\Psi^b\sigma^c\overline{\Psi}^d. \tag{41}$$

Then, the S -supersymmetry transformation turns into

此时，S 超对称变换变为

$$\delta_\eta e_m{}^a = 0,\ \delta_\eta\Psi_{m\alpha} = \left(\sigma_m\overline{\eta}\right)_\alpha,\ \delta_\eta\mathfrak{A}_m = \frac{3}{4}\mathrm{i}\left(\eta\Psi_m - \overline{\eta}\overline{\Psi}_m\right). \tag{42}$$

It turns out that the simplest version of the Q -supersymmetry transformation corresponds to the variation

可以证明，最简单形式的 Q 超对称变换对应如下变分

$$\hat{\delta}_\varepsilon = \delta_\varepsilon + \delta_{\eta(\varepsilon)} + \delta_{K(\varepsilon)} + \delta_{\varphi(\varepsilon)}, \tag{43a}$$

$$\eta^\alpha(\varepsilon) = -\mathrm{i}\left(\nabla_b\bar{\varepsilon}\tilde{\sigma}^b\right)^\alpha - \varepsilon^\alpha\overline{\Psi}^n\overline{\Psi}_n + \frac{1}{2}\left(\bar{\varepsilon}\tilde{\sigma}_a\right)^\alpha\Psi^n\sigma^a\overline{\Psi}_n$$

$$-\frac{\mathrm{i}}{4}\left(\bar{\varepsilon}\tilde{\sigma}_a\right)^\alpha\varepsilon^{abcd}\Psi_b\sigma_c\overline{\Psi}_d \tag{43b}$$

$$K_{\alpha\beta}(\varepsilon) = \frac{\mathrm{i}}{2}\left[\left(\sigma_n\overline{\Psi}^n\right)_\alpha\varepsilon_\beta + \left(\sigma_n\overline{\Psi}^n\right)_\beta\varepsilon_\alpha - \left(\sigma_n\bar{\varepsilon}\right)_\alpha\Psi^n_\beta - \left(\sigma_n\bar{\varepsilon}\right)_\beta\Psi^n_\alpha\right], \tag{43c}$$

$$\varphi(\varepsilon) = \frac{1}{2}\left(\Psi^n\sigma_n\bar{\varepsilon} + \varepsilon\sigma_n\overline{\Psi}^n\right). \tag{43d}$$

Then, we end up with the following transformation laws:

最终我们得到如下变换定律:

$$\hat{\delta}_\varepsilon e_m{}^a = \mathrm{i}\left(\varepsilon\sigma^a\overline{\Psi}_m - \Psi_m\sigma^a\bar{\varepsilon}\right), \tag{44a}$$

$$\hat{\delta}_\varepsilon\Psi_m = 2\widehat{\nabla}_m\varepsilon = 2\left(\nabla_m - \frac{1}{2}\widehat{\omega}_m{}^{bc}\sigma_{bc} + \mathrm{i}\mathfrak{A}_m\right)\varepsilon, \tag{44b}$$

$$\hat{\delta}_\varepsilon\mathfrak{A}_m = \frac{1}{2}\varepsilon\sigma^n\left(\widehat{\nabla}_m\overline{\Psi}_n - \widehat{\nabla}_n\overline{\Psi}_m\right) - \frac{1}{2}\left(\widehat{\nabla}_m\Psi_n - \widehat{\nabla}_n\Psi_m\right)\sigma^n\bar{\varepsilon}$$

$$-\frac{\mathrm{i}}{4}g_{mn}\varepsilon^{nijk}\left(\widehat{\nabla}_i\Psi_j\sigma_k\bar{\varepsilon} - \varepsilon\sigma_k\widehat{\nabla}_i\overline{\Psi}_j\right), \tag{44c}$$

where we have introduced the covariant derivative with torsion

其中我们引入了含挠协变导数

$$\widehat{\nabla}_m = \nabla_m - \frac{1}{2}\widehat{\omega}_m{}^{bc}M_{bc} - \mathrm{i}w\mathfrak{A}_m, \tag{45a}$$

$$\widehat{\omega}_{cab}=-\frac{1}{2}\left(\widehat{C}_{abc}+\widehat{C}_{acb}-\widehat{C}_{bca}\right),\ \widehat{C}_{abc}=\frac{\mathrm{i}}{2}\left(\Psi_a\sigma_c\overline{\Psi}_b-\Psi_b\sigma_c\overline{\Psi}_a\right), \tag{45b}$$

with w being the $\mathrm{U}(1)_R$ charge of a field Y with the $\mathrm{U}(1)_R$ transformation law $\mathrm{Y}\to \mathrm{e}^{\mathrm{i}w\varphi}\mathrm{Y}$

其中 w 是满足变换律 $\mathrm{Y}\to \mathrm{e}^{\mathrm{i}w\varphi}\mathrm{Y}$ 的场 Y 的 $\mathrm{U}(1)_R$ 荷

It follows from the above analysis that the gauge fields $\left\{e_m{}^a,\Psi_{m\alpha},\overline{\Psi}_m^{\dot\alpha},\mathfrak{A}_m\right\}$ form a multiplet under the local S - and Q -supersymmetry transformations. It will be referred to as the reduced Weyl multiplet (The Weyl multiplet also includes a dilatation gauge field, b_m , but this proves to describe purely gauge degrees of freedom (see, e.g., [28, 33] for reviews and section "Component Reduction and the Weyl Multiplet')).) of conformal supergravity, since it is a supersymmetric generalisation of conformal gravity, in which the gauge group includes the Weyl transformations $e_m{}^a(x)\to \mathrm{e}^{-\tau(x)}e_m{}^a(x)$.

由上述分析可知，规范场 $\left\{e_m{}^a,\Psi_{m\alpha},\overline{\Psi}_m^{\dot\alpha},\mathfrak{A}_m\right\}$ 在局域 S 和 Q 超对称变换下形成一个多重态。它被称为约化外尔多重态 (外尔多重态还包含一个伸缩规范场 b_m ，但可以证明该场仅描述规范自由度，相关综述参见例如 [28, 33]，也可参见“分量约化与外尔多重态”一节)，属于共形超引力，因为它是共形引力的超对称推广，而共形引力的规范群包含外尔变换 $e_m{}^a(x)\to \mathrm{e}^{-\tau(x)}e_m{}^a(x)$ 。

Conformal Superspace

共形超空间

In the previous section, we have reviewed a simple approach to obtain the reduced Weyl multiplet of conformal supergravity from superspace. In this setting, conformal supergravity is described by the gravitational superfield $\mathcal{H}^m$, which defines the embedding of curved superspace $\mathcal{M}^{4|4}(\mathcal{H})$ in the chiral superspace $\mathbb{C}^{4|2}$, Eq. (24). Although this approach is elegant and geometric, it is not covariant and does not offer powerful tools to construct manifestly gauge-invariant supergravity actions and to engineer general couplings of supergravity to matter. Such tools are provided by the so-called conformal superspace approach [32], which is reviewed in the present section.

在上一节中，我们回顾了从超空间得到共形超引力约化外尔多重态的简单方法。在此框架下，共形超引力由引力超场 $\mathcal{H}^m$ 描述，该超场定义了弯曲超空间 $\mathcal{M}^{4|4}(\mathcal{H})$ 在手征超空间 $\mathbb{C}^{4|2}$ 中的嵌入，见式 (24)。尽管该方法优雅且具有几何性，但它并不协变，也无法为构造明显规范不变的超引力作用量、设计超引力与物质的一般耦合提供有力工具。这类工具由所谓的共形超空间方法 [32] 提供，本文将在本节回顾该方法。

Gauging the Superconformal Algebra in Superspace

超空间中超共形代数的规范定域化

Conformal superspace is a gauge theory of the superconformal algebra. It can be identified with a pair $\left(\mathcal{M}^{4|4},\nabla\right)$. Here $\mathcal{M}^{4|4}$ denotes a supermanifold parametrised by local coordinates $z^M=\left(x^m,\theta^\mu,\overline{\theta}_{\dot\mu}\right)$, and ∇

is a covariant derivative associated with the superconformal algebra. We recall that the generators $X_{\tilde{a}}$ of the supercon-formal algebra are given by Eq. (11). They can be grouped in two disjoint subsets:

共形超空间是超共形代数的规范理论。它可表示为一对 $(\mathcal{M}^{4|4},\nabla)$。其中 $\mathcal{M}^{4|4}$ 表示由局部坐标 $z^M=\left(x^m,\theta^\mu,\bar{\theta}_{\dot{\mu}}\right)$ 参数化的超流形，∇ 是与超共形代数相关联的协变导数。回顾可知，超共形代数的生成元 $X_{\tilde{a}}$ 由式 (11) 给出，可分为两个不相交子集:

$$X_{\tilde{a}}=\left(P_A,X_{\underline{a}}\right),\ X_{\underline{a}}=\left(M_{ab},\mathbb{D},\mathbb{Y},K^A\right), \tag{46}$$

each of which constitutes a superalgebra:

每个子集各自构成一个超代数:

$$[P_A,P_B\}=-f_{AB}{}^{C}P_C, \tag{47a}$$

$$\left[X_{\underline{a}},X_{\underline{b}}\right\}=-f_{\underline{a}\underline{b}}{}^{\underline{c}}X_{\underline{c}}, \tag{47b}$$

$$\left[X_{\underline{a}},P_B\right\}=-f_{\underline{a}B}{}^{\underline{c}}X_{\underline{c}}-f_{\underline{a}B}{}^{C}P_C. \tag{47c}$$

Here the structure constants $f_{AB}{}^{C}$ contain only one nonzero component, which is $f_\alpha{}^{\dot{\beta}c}=2\mathrm{i}(\sigma^c)_\alpha{}^{\dot{\beta}}$.

此处结构常数 $f_{AB}{}^{C}$ 仅含一个非零分量，即 $f_\alpha{}^{\dot{\beta}c}=2\mathrm{i}(\sigma^c)_\alpha{}^{\dot{\beta}}$.

In order to define the covariant derivatives, $\nabla_A=\left(\nabla_a,\nabla_\alpha,\bar{\nabla}^{\dot{\alpha}}\right)$, we associate with each generator $X_{\underline{a}}=\left(M_{ab},\mathbb{D},\mathbb{Y},K^A\right)=\left(M_{ab},\mathbb{D},\mathbb{Y},K^a,S^\alpha,\bar{S}_{\dot{\alpha}}\right)$ a connection one-form $\omega^{\underline{a}}=\left(\Omega^{ab},B,\Phi,\mathfrak{F}_A\right)=\left(\Omega^{ab},B,\Phi,\mathfrak{F}_a,\mathfrak{F}_a,\bar{\mathfrak{F}}^{\dot{\alpha}}\right)=\mathrm{d}z^M\omega_M{}^{\underline{a}}$, and with P_A a supervielbein one-form $E^A=\left(E^a,E^\alpha,\bar{E}_{\dot{\alpha}}\right)=\mathrm{d}z^M E_M{}^A$ (the latter will be often referred to as the vielbein). It is assumed that the supermatrix $E_M{}^A$ is nonsingular, $E:=\operatorname{Ber}\left(E_M{}^A\right)\neq 0$, and hence there exists a unique inverse supervielbein. The latter is given by the supervector fields $E_A=E_A{}^M(z)\,\partial_M$, with $\partial_M=\partial/\partial z^M$, which constitute a new basis for the tangent space at each point $z^M\in\mathcal{M}^{4|4}$. The supermatrices $E_A{}^M$ and $E_M{}^A$ satisfy the properties $E_A{}^M E_M{}^B=\delta_A{}^B$ and $E_M{}^A E_A{}^N=\delta_M{}^N$. With respect to the basis E^A, the connection is expressed as $\omega^{\underline{a}}=E^B\omega_B{}^{\underline{a}}$, where $\omega_B{}^{\underline{a}}=E_B{}^M\omega_M{}^{\underline{a}}$. The covariant derivatives are given by

为了定义协变导数 $\nabla_A=\left(\nabla_a,\nabla_\alpha,\bar{\nabla}^{\dot{\alpha}}\right)$，我们将每个生成元 $X_{\underline{a}}=\left(M_{ab},\mathbb{D},\mathbb{Y},K^A\right)=\left(M_{ab},\mathbb{D},\mathbb{Y},K^a,S^\alpha,\bar{S}_{\dot{\alpha}}\right)$ 对应一个联络 1-形式 $\omega^{\underline{a}}=\left(\Omega^{ab},B,\Phi,\mathfrak{F}_A\right)=\left(\Omega^{ab},B,\Phi,\mathfrak{F}_a,\mathfrak{F}_a,\bar{\mathfrak{F}}^{\dot{\alpha}}\right)=\mathrm{d}z^M\omega_M{}^{\underline{a}}$，将 P_A 对应一个超 Vielbein 1-形式 $E^A=\left(E^a,E^\alpha,\bar{E}_{\dot{\alpha}}\right)=\mathrm{d}z^M E_M{}^A$ (后者常简称为 Vielbein)。假设超矩阵 $E_M{}^A$ 非奇异 $E:=\operatorname{Ber}\left(E_M{}^A\right)\neq 0$，因此存在唯一的逆超 Vielbein。逆超 Vielbein 由超向量场 $E_A=E_A{}^M(z)\,\partial_M$ 给出，满足 $\partial_M=\partial/\partial z^M$，它们构成每一点 $z^M\in\mathcal{M}^{4|4}$ 处切空间的一组新基。超矩阵 $E_A{}^M$ 与 $E_M{}^A$ 满足性质 $E_A{}^M E_M{}^B=\delta_A{}^B$ 和 $E_M{}^A E_A{}^N=\delta_M{}^N$。在基 E^A 下，联络可表示为 $\omega^{\underline{a}}=E^B\omega_B{}^{\underline{a}}$，其中 $\omega_B{}^{\underline{a}}=E_B{}^M\omega_M{}^{\underline{a}}$。协变导数由下式给出

$$\nabla_A=E_A-\omega_A{}^{\underline{b}}X_{\underline{b}}=E_A-\frac{1}{2}\Omega_A{}^{bc}M_{bc}-B_A\mathbb{D}-\mathrm{i}\Phi_A\mathbb{Y}-\mathfrak{F}_{AB}K^B. \tag{48}$$

They can be recast in terms of one-forms:

它们可以用 1-形式重写为:

$$\nabla = \mathrm{d} - \omega^{\underline{a}} X_{\underline{a}},\ \nabla = E^A \nabla_A. \tag{49}$$

The translation generators P_B do not show up in (48) and (49). It is assumed that the operators ∇_A replace P_A and obey the graded commutation relations

平移生成元 P_B 没有出现在 (48) 和 (49) 中。假设算符 ∇_A 替代 P_A ，并满足分次对易关系

$$\left[X_{\underline{b}}, \nabla_A\right\} = -f_{\underline{b}A}{}^C \nabla_C - f_{\underline{b}A}{}^{\underline{c}} X_{\underline{c}}, \tag{50}$$

compare with (47c). In particular, the algebra of K^A with ∇_B is given by

与 (47c) 对比。特别地，K^A 与 ∇_B 的代数由下式给出

$$[K^a, \nabla_b] = 2\delta_b^a \mathbb{D} + 2M^a{}_b, \tag{51a}$$

$$\left\{S^\alpha, \nabla_\beta\right\} = \delta_\beta^\alpha (2\mathbb{D} - 3\mathbb{Y}) - 4M^\alpha{}_\beta, \tag{51b}$$

$$\left\{\bar{S}_{\dot\alpha}, \bar\nabla^{\dot\beta}\right\} = \delta_{\dot\alpha}^{\dot\beta} (2\mathbb{D} + 3\mathbb{Y}) + 4\bar{M}_{\dot\alpha}{}^{\dot\beta}, \tag{51c}$$

$$\left[K^a, \nabla_\beta\right] = -\mathrm{i}(\sigma^a)_\beta{}^{\dot\beta} \bar{S}_{\dot\beta},\ \left[K^a, \bar\nabla^{\dot\beta}\right] = -\mathrm{i}(\sigma^a)^{\dot\beta}{}_\beta S^\beta, \tag{51d}$$

$$[S^\alpha, \nabla_b] = \mathrm{i}(\sigma_b)^\alpha{}_{\dot\beta} \bar\nabla^{\dot\beta},\ \left[\bar{S}_{\dot\alpha}, \nabla_b\right] = \mathrm{i}(\sigma_b)_{\dot\alpha}{}^\beta \nabla_\beta, \tag{51e}$$

where all other graded commutators vanish.

其中所有其他分次对易子均为零。

By definition, the gauge group of conformal supergravity is generated by local transformations of the form

根据定义，共形超引力的规范群由如下形式的局域变换生成

$$\delta_{\mathcal{K}} \nabla_A = [\mathcal{K}, \nabla_A] \tag{52a}$$

$$\mathcal{K} = \xi^B \nabla_B + \Lambda^{\underline{b}} X_{\underline{b}} = \xi^B \nabla_B + \frac{1}{2} K^{bc} M_{bc} + \sum \mathbb{D} + \mathrm{i}\rho \mathbb{Y} + \Lambda_B K^B, \tag{52b}$$

where the gauge parameters satisfy natural reality conditions. In applying Eq. (52), we interpret that

其中规范参数满足自然实性条件。应用式 (52) 时，我们将其解释为

$$\nabla_A \xi^B \;:=\; E_A \xi^B + \omega_A{}^{\underline{c}} \xi^D f_{D\underline{c}}{}^B, \tag{53a}$$

$$\nabla_A \Lambda^{\underline{b}} \;:=\; E_A \Lambda^{\underline{b}} + \omega_A{}^{\underline{c}} \xi^D f_{D\underline{c}}{}^{\underline{b}} + \omega_A{}^{\underline{c}} \Lambda^{\underline{d}} f_{\underline{dc}}{}^{\underline{b}}. \tag{53b}$$

Then, it follows from (52) that

那么由 (52) 可推得

$$\delta_{\mathcal{K}} E^A = \mathrm{d}\xi^A + E^B \Lambda^{\underline{c}} f_{\underline{c}B}{}^A + \omega^{\underline{b}} \xi^C f_{C\underline{b}}{}^A + E^B \xi^C \mathcal{T}_{CB}{}^A, \tag{54a}$$

$$\delta_{\mathcal{K}} \omega^{\underline{a}} = \mathrm{d}\Lambda^{\underline{a}} + \omega^{\underline{b}} \Lambda^{\underline{c}} f_{\underline{cb}}{}^{\underline{a}} + \omega^{\underline{b}} \xi^C f_{C\underline{b}}{}^{\underline{a}} + E^B \Lambda^{\underline{c}} f_{\underline{c}B}{}^{\underline{a}} + E^B \xi^C \mathcal{R}_{CB}{}^{\underline{a}}.$$

(54b)

Here we have made use of the graded commutation relations:

此处我们用到了如下分次对易关系:

$$[\nabla_A, \nabla_B\} = -\mathcal{T}_{AB}{}^C \nabla_C - \mathcal{R}_{AB}{}^{\underline{c}} X_{\underline{c}}, \tag{55}$$

where $\mathcal{T}_{AB}{}^C$ and $\mathcal{R}_{AB}{}^c$ denote the torsion and the curvature, respectively. They can be recast in terms of two-forms:

其中 $\mathcal{T}_{AB}{}^C$ 和 $\mathcal{R}_{AB}{}^c$ 分别表示挠率和曲率。它们可以用 2-形式重写为:

$$\mathcal{T}^A \;:=\; \frac{1}{2} E^C \wedge E^B \mathcal{T}_{BC}{}^A = \mathrm{d}E^A - E^C \wedge \omega^{\underline{b}} f_{\underline{b}C}{}^A, \tag{56a}$$

$$\mathcal{R}^{\underline{a}} \;:=\; \frac{1}{2} E^C \wedge E^B \mathcal{R}_{BC}{}^{\underline{a}} = \mathrm{d}\omega^{\underline{a}} - E^C \wedge \omega^{\underline{b}} f_{\underline{b}C}{}^{\underline{a}} - \frac{1}{2} \omega^{\underline{c}} \wedge \omega^{\underline{b}} f_{\underline{bc}}{}^{\underline{a}}. \tag{56b}$$

Making use of the graded Jacobi identity:

利用分次雅可比恒等式:

$$0 = (-1)^{\varepsilon_{\underline{a}} \varepsilon_C} \left[X_{\underline{a}}, [\nabla_B, \nabla_C\}\right\} + \text{ (two cycles)} \tag{57}$$

we derive the action of $X_{\underline{a}}$ on the geometric objects

我们推导出了 $X_{\underline{a}}$ 在几何对象上的作用

$$X_{\underline{a}} \mathcal{T}_{BC}{}^D = -(-1)^{\varepsilon_{\underline{a}}(\varepsilon_B + \varepsilon_C)} \mathcal{T}_{BC}{}^E f_{E\underline{a}}{}^D - 2 f_{\underline{a}[B}{}^E \mathcal{T}_{|E|C\}}{}^D - 2 f_{\underline{a}[B}{}^e f_{|\underline{e}|C]}{}^D,$$

(58a)

$$X_{\underline{a}} \mathcal{R}_{BC}{}^{\underline{d}} = -(-1)^{\varepsilon_{\underline{a}}(\varepsilon_B + \varepsilon_C)} \left(\mathcal{T}_{BC}{}^E f_{E\underline{a}}{}^{\underline{d}} + \mathcal{R}_{BC}{}^{\underline{e}} f_{\underline{ea}}{}^{\underline{d}}\right) - 2 f_{\underline{a}[B}{}^E \mathcal{R}_{|E|C\}}{}^{\underline{d}}$$

$$-2f_{\underline{a}[B}{}^{\underline{e}}f_{|\underline{e}|C]}{}^{\underline{d}}. \tag{58b}$$

The supergravity gauge group acts on a conformal tensor superfield U (with suppressed indices) as

超引力规范群作用在共形张量超场 U (指标已省略) 上的形式为

$$\delta_{\mathcal{K}}U = \mathcal{K}U. \tag{59}$$

The torsion $\mathcal{T}_{AB}{}^{C}$ and the curvature $\mathcal{R}_{AB}{}^{c}$ are conformal tensor superfields, for which the action of the generators $X_{\underline{a}}$ is specified by the relations (58). Of special significance are primary superfields. A tensor superfield U (with suppressed indices) is said to be primary if it is characterised by the properties

挠率 $\mathcal{T}_{AB}{}^{C}$ 和曲率 $\mathcal{R}_{AB}{}^{c}$ 是共形张量超场，生成元 $X_{\underline{a}}$ 对其的作用由关系式 (58) 确定。其中本原超场具有特殊意义。若一个张量超场 U (指标已省略) 满足下述性质，则称它是本原的

$$K^{A}U = 0,\ \mathbb{D}U = wU,\ \mathbb{Y}U = cU, \tag{60}$$

for some real constants w and c which are called the dimension (or Weyl weight) and $\mathrm{U}(1)_{R}$ charge of U , respectively. It follows from (19b) that if a superfield is annihilated by the S -supersymmetry generators, then it is necessarily primary.

其中 w 和 c 是实常数，分别被称为 U 的维数 (或外尔权重) 和 $\mathrm{U}(1)_{R}$ 荷。由 (19b) 可得，若一个超场被 S 超对称生成元零化，则它必然是本原的。

Let us summarise some important features of the gauging procedure. In curved superspace, the superconformal algebra (47) is replaced with

我们来总结规范过程的若干重要性质。在弯曲超空间中，超共形代数 (47) 被替换为

$$\left[X_{\underline{a}}, X_{\underline{b}}\right\} = -f_{\underline{a}\underline{b}}{}^{\underline{c}}X_{\underline{c}}, \tag{61a}$$

$$\left[X_{\underline{a}}, \nabla_{B}\right\} = -f_{\underline{a}B}{}^{C}\nabla_{C} - f_{\underline{a}B}{}^{\underline{c}}X_{\underline{c}}, \tag{61b}$$

$$[\nabla_{A}, \nabla_{B}\} = -\mathcal{T}_{AB}{}^{C}\nabla_{C} - \mathcal{R}_{AB}{}^{\underline{c}}X_{\underline{c}}. \tag{61c}$$

Here the torsion and curvature tensors obey Bianchi identities, which follow from

此处挠率和曲率张量满足比安基恒等式，该恒等式由下式导出

$$0 = (-1)^{\varepsilon_{A}\varepsilon_{C}}\left[\nabla_{A}, [\nabla_{B}, \nabla_{C}\}\right\} + \text{ (two cycles)}\,. \tag{62}$$

Unlike (47), which is determined by the structure constants, the graded commutation relations (61) involve structure functions $\mathcal{T}_{AB}{}^{C}$ and $\mathcal{R}_{AB}{}^{c}$. Such an algebraic structure is sometimes called a soft algebra (see, e.g., [28]).

与由结构常数确定的 (47) 不同，分次对易关系 (61) 包含结构函数 $\mathcal{T}_{AB}{}^{C}$ 和 $\mathcal{R}_{AB}{}^{c}$ 。这种代数结构有时被称为软代数 (例如参见文献 [28])。

Conventional Constraints for Weyl Multiplet

外尔多重态的常规约束

The framework described in the previous subsection defines a geometric setup to obtain a multiplet of conformal supergravity containing the metric. However, in general, the resulting multiplet is reducible. To obtain the irreducible multiplet described in section ”Rigid and Local Superconformal Transformations,” it is necessary to impose constraints on the torsion and curvatures appearing in Eq. (55). This is a standard task in geometric superspace approaches to supergravity, and it is pedagogically reviewed in [13,35]. One beautiful feature of the construction of [32] is the simplicity of the superspace constraints needed to obtain the Weyl multiplet of conformal supergravity. In fact, to obtain a sufficient set of constraints, one requires the algebra (55) to have a Yang-Mills structure. Specifically, following [32], one imposes

上一小节描述的框架给出了构造包含度规的共形超引力多重态的几何设置, 但一般而言得到的多重态是可约的。为了得到章节“刚性与局域超共形变换”中描述的不可约多重态，需要对式 (55) 中的挠率与曲率施加约束。这是超引力的几何超空间方法中的标准步骤，在文献 [13,35] 中有教学性回顾。文献 [32] 构造的一个优美特点是，得到共形超引力外尔多重态所需的超空间约束十分简单。事实上，要得到一组充分约束，只需要求代数 (55) 具有杨-米尔斯结构。具体来说，遵循文献 [32] 的工作，我们施加

$$\left\{\nabla_\alpha, \nabla_\beta\right\} = 0,\ \left\{\overline{\nabla}_{\dot\alpha}, \overline{\nabla}_{\dot\beta}\right\} = 0,\ \left\{\nabla_\alpha, \overline{\nabla}_{\dot\alpha}\right\} = -2\mathrm{i}\nabla_{\alpha\dot\alpha}, \tag{63a}$$

$$\left[\nabla_\alpha, \nabla_{\beta\dot\beta}\right] = 2\mathrm{i}\varepsilon_{\alpha\beta}\bar{\mathcal{W}}_{\dot\beta},\ \left[\overline{\nabla}_{\dot\alpha}, \nabla_{\beta\dot\beta}\right] = -2\mathrm{i}\varepsilon_{\dot\alpha\dot\beta}\mathcal{W}_\beta, \tag{63b}$$

where the operator $\mathcal{W}_{\dot\alpha}$ is the complex conjugate of $\mathcal{W}_\alpha$. The latter takes the form

其中算符 $\mathcal{W}_{\dot\alpha}$ 是 $\mathcal{W}_\alpha$ 的复共轭。后者形式为

$$\mathcal{W}_\alpha = \frac{1}{2}\mathcal{W}(M)_\alpha{}^{cd}M_{cd} + \mathcal{W}(\mathbb{D})_\alpha\mathbb{D} + \mathrm{i}\mathcal{W}(\mathbb{Y})_\alpha\mathbb{Y}$$

$$+\mathcal{W}(S)_{\alpha\gamma}S^\gamma + \mathcal{W}(S)_\alpha{}^{\dot\gamma}\bar{S}_{\dot\gamma} + \mathcal{W}(K)_{\alpha c}K^c. \tag{64}$$

Having imposed the constraints (63), the Bianchi identities (62) become nontrivial and now play the role of consistency conditions, which may be used to determine the torsion and curvature. Their solution is as follows:

施加约束 (63) 后，比安基恒等式 (62) 变得非平凡，现在起到相容性条件的作用，可以用它确定挠率与曲率。其解如下:

$$\left[\nabla_\alpha, \nabla_{\beta\dot\beta}\right] = \mathrm{i}\varepsilon_{\alpha\beta}\left(2\bar W_{\dot\beta\dot\gamma\dot\delta}\bar M^{\dot\gamma\dot\delta} - \frac{1}{2}\overline{\nabla}^{\dot\alpha}\bar W_{\dot\alpha\dot\beta\dot\gamma}\bar S^{\dot\gamma} + \frac{1}{2}\nabla_\gamma{}^{\dot\alpha}\bar W_{\dot\alpha\dot\beta\dot\gamma}K^{\gamma\dot\gamma}\right), \tag{65a}$$

$$\left[\overline{\nabla}_{\dot\alpha}, \nabla_{\beta\dot\beta}\right] = -\mathrm{i}\varepsilon_{\dot\alpha\dot\beta}\left(2W_\beta{}^{\gamma\delta}M_{\gamma\delta} + \frac{1}{2}\nabla^\alpha W_{\alpha\beta\gamma}S^\gamma + \frac{1}{2}\nabla^{\alpha\dot\gamma}W_{\alpha\beta}{}^\gamma K_{\gamma\dot\gamma}\right), \tag{65b}$$

$$\left[\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}\right] = \varepsilon_{\dot\alpha\dot\beta}\psi_{\alpha\beta} + \varepsilon_{\alpha\beta}\overline{\psi}_{\dot\alpha\dot\beta}, \tag{65c}$$

where the symmetric bispinor operator $\psi_{\alpha\beta}$ and its conjugate $\overline{\psi}_{\dot\alpha\dot\beta}$ are given by

其中对称双旋量算符 $\psi_{\alpha\beta}$ 及其共轭 $\overline{\psi}_{\dot\alpha\dot\beta}$ 由下式给出

$$\psi_{\alpha\beta} = W_{\alpha\beta}{}^\gamma\nabla_\gamma + \nabla^\gamma W_{\alpha\beta}{}^\delta M_{\gamma\delta} - \frac{1}{8}\nabla^2 W_{\alpha\beta\gamma}S^\gamma + \frac{\mathrm{i}}{2}\nabla^{\gamma\dot\gamma}W_{\alpha\beta\gamma}\bar S_{\dot\gamma}$$

$$+\frac{1}{4}\nabla^{\gamma\dot\delta}\nabla_{(\alpha}W_{\beta)\gamma}{}^\delta K_{\delta\dot\delta} + \frac{1}{2}\nabla^\gamma W_{\alpha\beta\gamma}\mathbb{D} - \frac{3}{4}\nabla^\gamma W_{\alpha\beta\gamma}\mathbb{Y} \tag{65d}$$

$$\overline{\psi}_{\dot\alpha\dot\beta} = -\bar W_{\dot\alpha\dot\beta}{}^{\dot\gamma}\overline{\nabla}_{\dot\gamma} - \overline{\nabla}^{\dot\gamma}\bar W_{\dot\alpha\dot\beta}{}^{\dot\delta}\bar M_{\dot\gamma\dot\delta} + \frac{1}{8}\overline{\nabla}^2\bar W_{\dot\alpha\dot\beta\dot\gamma}\bar S^{\dot\gamma} + \frac{\mathrm{i}}{2}\nabla^{\gamma\dot\gamma}\bar W_{\dot\alpha\dot\beta\dot\gamma}S_\gamma$$

$$-\frac{1}{4}\nabla^{\delta\dot\gamma}\overline{\nabla}_{(\dot\alpha}\bar W_{\dot\beta)\dot\gamma}{}^{\dot\delta}K_{\delta\dot\delta} - \frac{1}{2}\overline{\nabla}^{\dot\gamma}\bar W_{\dot\alpha\dot\beta\dot\gamma}\mathbb{D} - \frac{3}{4}\overline{\nabla}^{\dot\gamma}\bar W_{\dot\alpha\dot\beta\dot\gamma}\mathbb{Y}. \tag{65e}$$

The structure of the conformal superspace algebra leads to highly nontrivial implications. In particular, Eq. (51c) implies that primary covariantly chiral superfields can carry only undotted spinor indices. Given such a superfield, $\phi_{\alpha(n)}$, Eq. (51c) further implies that the $\mathrm{U}(1)_R$ charge of $\phi_{\alpha(n)}$ is determined in terms of its dimension,

共形超空间代数的结构具有高度非平凡的推论。特别地，式 (51c) 表明，基本协变手征超场只能带无点旋量指标。对于这样的超场 $\phi_{\alpha(n)}$ ，式 (51c) 进一步推出 $\phi_{\alpha(n)}$ 的 $\mathrm{U}(1)_R$ 荷由其维度确定，

$$K^B\phi_{\alpha(n)} = 0,\ \overline{\nabla}^{\dot\beta}\phi_{\alpha(n)} = 0,\ \mathbb{D}\phi_{\alpha(n)} = w\phi_{\alpha(n)} \Rightarrow c = -\frac{2}{3}w. \tag{66}$$

There is a regular procedure to construct such constrained multiplets. Given a complex tensor superfield $\psi_{\alpha(n)}$ with the superconformal properties

存在构造这类约束多重态的正则程序。给定满足超共形性质的复张量超场 $\psi_{\alpha(n)}$

$$K^B\psi_{\alpha(n)} = 0,\ \mathbb{D}\psi_{\alpha(n)} = (w-1)\,\psi_{\alpha(n)},\ \mathbb{Y}\psi_{\alpha(n)} = 2\left(1-\frac{1}{3}w\right)\psi_{\alpha(n)}, \tag{67}$$

its descendant

其后代

$$\phi_{\alpha(n)} = -\frac{1}{4}\overline{\nabla}^2\psi_{\alpha(n)} \tag{68}$$

proves to be primary and covariantly chiral. Here $\phi_{\alpha(n)}$ is invariant under gauge transformations of the form $\delta\psi_{\alpha(n)} = \overline{\nabla}_{\dot\beta}\lambda_{\alpha(n)}{}^{\dot\beta}$, where the gauge parameter $\lambda_{\alpha(n)}{}^{\dot\beta}$ is primary.

可证明是基本协变手征超场。此处 $\phi_{\alpha(n)}$ 在形如 $\delta\psi_{\alpha(n)} = \overline{\nabla}_{\dot\beta}\lambda_{\alpha(n)}{}^{\dot\beta}$ 的规范变换下不变，其中规范参数 $\lambda_{\alpha(n)}{}^{\dot\beta}$ 是基本的。

We note that the conformal superspace algebra is expressed in terms of a single superfield $W_{\alpha\beta\gamma} = W_{(\alpha\beta\gamma)}$, its conjugate $\bar{W}_{\dot\alpha\dot\beta\dot\gamma}$, and their covariant derivatives. This superfield defines an $\mathcal{N} = 1$ extension of the Weyl tensor; it is known as the super-Weyl tensor. Further, it is a primary chiral superfield of dimension 3/2

我们注意到，共形超空间代数可以用单个超场 $W_{\alpha\beta\gamma} = W_{(\alpha\beta\gamma)}$、其共轭 $\bar{W}_{\dot\alpha\dot\beta\dot\gamma}$ 以及它们的协变导数表示。这个超场定义了外尔张量的 $\mathcal{N} = 1$ 扩张，称为超外尔张量。此外，它是维度为 3/2 的基本手征超场

$$K^D W_{\alpha\beta\gamma} = 0,\ \overline{\nabla}^{\dot\beta} W_{\alpha\beta\gamma} = 0,\ \mathbb{D}W_{\alpha\beta\gamma} = \frac{3}{2}W_{\alpha\beta\gamma}, \tag{69}$$

and it obeys the Bianchi identity

且满足比安基恒等式

$$\mathcal{B}_{\alpha\dot\alpha} := \mathrm{i}\nabla^{\beta}{}_{\dot\alpha}\nabla^{\gamma}W_{\alpha\beta\gamma} = \mathrm{i}\nabla_{\alpha}{}^{\dot\beta}\overline{\nabla}^{\dot\gamma}\bar{W}_{\dot\alpha\dot\beta\dot\gamma} = \bar{\mathcal{B}}_{\alpha\dot\alpha}. \tag{70}$$

Here we have defined the superfield $\mathcal{B}_{\alpha\dot\alpha}$, which is the $\mathcal{N} = 1$ supersymmetric generalisation of the Bach tensor introduced in [38] (see also [39]). The super-Bach tensor, $\mathcal{B}_{\alpha\dot\alpha}$, proves to be primary, $K^B\mathcal{B}_{\alpha\dot\alpha} = 0$; carries weight 3, $\mathbb{D}\mathcal{B}_{\alpha\dot\alpha} = 3\mathcal{B}_{\alpha\dot\alpha}$; and satisfies the conservation equation

此处我们定义了超场 $\mathcal{B}_{\alpha\dot\alpha}$，它是文献 [38] 中引入的巴赫张量的 $\mathcal{N} = 1$ 超对称推广 (也可参见 [39])。超巴赫张量 $\mathcal{B}_{\alpha\dot\alpha}$ 可证明是基本的，$K^B\mathcal{B}_{\alpha\dot\alpha} = 0$；带权重 3，$\mathbb{D}\mathcal{B}_{\alpha\dot\alpha} = 3\mathcal{B}_{\alpha\dot\alpha}$；且满足守恒方程

$$\nabla^{\alpha}\mathcal{B}_{\alpha\dot\alpha} = 0 \Leftrightarrow \overline{\nabla}^{\dot\alpha}\mathcal{B}_{\alpha\dot\alpha} = 0. \tag{71}$$

At this stage, it is still necessary to show that the conformal superspace geometry described in this subsection indeed encodes the Weyl multiplet of conformal supergravity. We will demonstrate this in two different ways. First, we will describe the procedure of reducing the results of this section to their component field description. Second, we will prove that this geometry is equivalent to the U (1) , and consequently the Grimm-Wess-Zumino, superspace descriptions of conformal supergravity [29, 31]. It should also be mentioned that the Bianchi identities of conformal superspace, and in particular (69), can be solved in term of the conformal supergravity prepotential $\mathcal{H}^a$ described in section "Rigid and Local Supercon-formal Transformations." The prepotential description of conformal superspace can be found in [32]. In section "Supergravity Prepotentials" we will review the prepotential description of the Grimm-Wess-Zumino superspace geometry.

目前仍需证明，本小节描述的共形超空间几何确实蕴含共形超引力的 Weyl 多重态。我们将通过两种不同方式证明: 首先，我们会介绍将本节结果约化为分量场描述的步骤；其次，我们将证明该几何等价于 U (1) ，也就是格里姆-韦斯-祖米诺共形超引力的超空间描述 [29, 31]。还需要指出的是，共形超空间的比安基恒等式，尤其是式 (69)，可以用“刚性与局域超共形变换”一节中描述的共形超引力预势 $\mathcal{H}^a$ 求解。共形超空间的预势描述可参考文献 [32]。我们将在“超引力预势”一节回顾格里姆-韦斯-祖米诺超空间几何的预势描述。

Superconformal Action Principles

超共形作用量原理

In order to formulate locally superconformal field theories, an action principle is required. As in the rigid supersymmetric case, locally superconformal actions can be constructed in two different ways: either as integrals over the full superspace or over its chiral subspace. Here we consider separately these two options.

为了构建局域超共形场论，我们需要作用量原理。与刚性超对称的情况类似，局域超共形作用量可通过两种不同方式构造: 要么是对整个超空间积分，要么是对其手征子空间积分。本文我们将分别讨论这两种方案。

We look for a scalar superfield $\mathcal{L}$ such that the action

我们寻找满足条件的标量超场 $\mathcal{L}$ ，使得作用量

$$S = \int \mathrm{d}^{4|4} z E \mathcal{L}, \ \mathrm{d}^{4|4} z \ := \ \mathrm{d}^4 x \ \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \tag{72}$$

is locally superconformal. Performing a gauge transformation, Eqs. (52) and (59), we arrive at the variation

是局域超共形的。进行规范变换，利用式 (52) 和 (59)，我们得到变分

$$\delta_{\mathcal{K}} S = \int \mathrm{d}^{4|4} z E \Big((-1)^{\varepsilon_A} \left[\nabla_A \left(\xi^A \mathcal{L} \right) + \xi^B \mathcal{T}_{BA}{}^A \mathcal{L} \right]$$

$$+ \Lambda^{\underline{b}} \left[(-1)^{\varepsilon_A} f_{\underline{b}A}{}^A \mathcal{L} + X_{\underline{b}} \mathcal{L} \right] \Big), \tag{73}$$

which must vanish. Now, requiring the contributions containing Λ^b to vanish gives

该变分必须为零。现在，要求含 Λ^b 的贡献为零可得

$$X_{\underline{b}} \mathcal{L} = -(-1)^{\varepsilon_A} f_{\underline{b}A}{}^A \mathcal{L} \ \Leftrightarrow \ K^B \mathcal{L} = 0, \ \mathbb{D} \mathcal{L} = 2 \mathcal{L}, \ \mathbb{Y} \mathcal{L} = 0.$$

(74)

Once the conditions (74) are satisfied, it is readily seen that the remaining ξ - dependent contributions in (73) cancel out. In summary, given a primary real dimension-2 scalar Lagrangian $\mathcal{L}$, the action (72) is locally superconformal.

一旦条件 (74) 得到满足，不难发现式 (73) 中其余依赖 ξ 的贡献会相互抵消。综上，对于满足要求的、维度为 2 的原生实标量拉格朗日量 $\mathcal{L}$ ，作用量 (72) 是局域超共形的。

Given a primary chiral scalar Lagrangian $\mathcal{L}_{\mathrm{c}}$ of weight +3,

给定权为 +3 的原生手征标量拉格朗日量 $\mathcal{L}_{\mathrm{c}}$,

$$K^B\mathcal{L}_{\mathrm{c}} = 0,\ \overline{\nabla}_{\dot{\alpha}}\mathcal{L}_{\mathrm{c}} = 0,\ \mathbb{D}\mathcal{L}_{\mathrm{c}} = 3\mathcal{L}_{\mathrm{c}}, \tag{75}$$

the chiral action

该手征作用量

$$S_{\mathrm{c}} = \int \mathrm{d}^4x\ \mathrm{d}^2\theta\mathcal{E}\mathcal{L}_{\mathrm{c}} \tag{76}$$

is locally superconformal. Here $\mathcal{E}$ is a chiral density. The precise definition of $\mathcal{E}$ requires the use of a prepotential formulation for supergravity (see section "Chiral Action").

是局域超共形的。此处 $\mathcal{E}$ 是手征密度。对 $\mathcal{E}$ 的精确定义需要使用超引力的预势公式 (见“手征作用量”小节)。

A different definition of S_{c} exists, which is based on the use of a complex superfield Υ with the following superconformal properties (for some constant Δ):

S_{c} 还存在另一种定义，该定义基于具有如下超共形性质的复标量超场 Υ (对某个常数 Δ):

$$K^B\Upsilon = 0,\ \mathbb{D}\Upsilon = (\Delta - 1)\Upsilon,\ \mathbb{Y}\Upsilon = 2\left(1 - \frac{1}{3}\Delta\right)\Upsilon, \tag{77}$$

such that $\overline{\nabla}^2\Upsilon$ is nowhere vanishing, that is, $\left(\overline{\nabla}^2\Upsilon\right)^{-1}$ exists. Specifically, the chiral action may be identified with the functional

要求 $\overline{\nabla}^2\Upsilon$ 处处非零，即 $\left(\overline{\nabla}^2\Upsilon\right)^{-1}$ 存在。具体来说，手征作用量可以表示为该泛函

$$S_{\mathrm{c}} = -4\int \mathrm{d}^{4|4}zE\frac{\Upsilon}{\overline{\nabla}^2\Upsilon}\mathcal{L}_{\mathrm{c}}, \tag{78}$$

which possesses the two fundamental properties: (i) it is locally superconformal under the conditions (75), and (ii) it is independent of Υ ,

它具备两个基本性质:(i) 在条件 (75) 下是局域超共形的；(ii) 不依赖于 Υ ,

$$\delta_{\mathrm{Y}} \int \mathrm{d}^{4|4} z E \frac{\mathrm{Y}}{\bar{\nabla}^2 \mathrm{Y}} \mathcal{L}_{\mathrm{c}} = 0, \tag{79}$$

for an arbitrary variation $\delta \mathrm{Y}$. Using the representation (78) for the chiral action (76), it holds that

对任意变分 $\delta \mathrm{Y}$ 都成立。利用表示式 (78) 描述手征作用量 (76)，可得

$$\int \mathrm{d}^{4|4} z E \mathcal{L} = \int \mathrm{d}^4 x \ \mathrm{d}^2 \theta E \mathcal{L}_{\mathrm{c}}, \ \mathcal{L}_{\mathrm{c}} = -\frac{1}{4} \bar{\nabla}^2 \mathcal{L}. \tag{80}$$

This result can also be obtained using superspace normal coordinates [32] (see also [40]).

该结果也可以通过超空间正规坐标得到 [32](另见文献 [40])。

There is an alternative definition of the chiral action that follows from the superform approach to the construction of supersymmetric invariants [41,42]. It is based on the use of the following super four-form:

手征作用量还有另一种定义，源自构造超对称不变量的超形式方法 [41,42]，它基于如下四超形式:

$$\begin{aligned}\Xi_4 = 2\mathrm{i} \bar{E}_{\dot{\delta}} \wedge \bar{E}_{\dot{\gamma}} \wedge E^b \wedge E^a (\tilde{\sigma}_{ab})^{\dot{\gamma}\dot{\delta}} \mathcal{L}_{\mathrm{c}} + \frac{\mathrm{i}}{6} \varepsilon_{abcd} \bar{E}_{\dot{\delta}} \wedge E^c \wedge E^b \wedge E^a \big(\tilde{\sigma}^d\big)^{\dot{\delta}\delta} \nabla_\delta \mathcal{L}_{\mathrm{c}} \\ -\frac{1}{96} \varepsilon_{abcd} E^d \wedge E^c \wedge E^b \wedge E^a \nabla^2 \mathcal{L}_{\mathrm{c}}\end{aligned} \tag{81}$$

which was constructed by Binétruy et al. [43] and independently by Gates et al. [42] in the GWZ superspace. This superform is closed:

该超形式由 Binétruy 等人 [43] 在 GWZ 超空间中构造，Gates 等人 [42] 也独立得到了它。该超形式是闭的:

$$\mathrm{d}\, \Xi_4 = 0 \, . \tag{82}$$

It proves to be primary (The superform may be degauged to the GWZ superspace described in the next section. Then, the condition (83) is equivalent to the super-Weyl invariance of Ξ_4 . The latter property was proved in [44].)

事实证明它是本原的 (超形式可以退规范到下一节描述的 GWZ 超空间。此时条件 (83) 等价于 Ξ_4 的超魏尔不变性，该性质已在文献 [44] 中证明。)

$$K^B \Xi_4 = 0. \tag{83}$$

The chiral action (76) can be recast as an integral of Ξ_4 over a spacetime $\mathcal{M}^4$,

手征作用量 (76) 可以改写为 Ξ_4 在时空 $\mathcal{M}^4$ 上的积分，

$$S_{\mathrm{c}} = \int_{\mathcal{M}^4} \Xi_4 \tag{84a}$$

where $\mathcal{M}^4$ is the bosonic body of the curved superspace $\mathcal{M}^{4|4}$ obtained by switching off the Grassmann variables. It turns out that (84a) leads to the representation

其中 $\mathcal{M}^4$ 是关闭格拉斯曼变量后得到的弯曲超空间 $\mathcal{M}^{4|4}$ 的玻色主体。可以发现 (84a) 给出了如下表示

$$S_c = \int \mathrm{d}^4 x e \left(-\frac{1}{4}\nabla^2 + \frac{\mathrm{i}}{2}(\tilde{\sigma}^a)^{\dot{\alpha}\alpha}\overline{\Psi}_{a\dot{\alpha}}\nabla_\alpha - \left(\tilde{\sigma}^{ab}\right)^{\dot{\alpha}\dot{\beta}}\overline{\Psi}_{a\dot{\alpha}}\overline{\Psi}_{b\dot{\beta}}\right)\mathcal{L}_c\Big|_{\theta=0} \tag{84b}$$

which is the simplest way to reduce the action from superfields to components. Here $\overline{\Psi}_{a\dot{\alpha}} = e_a{}^m \overline{\Psi}_{m\dot{\alpha}}$ is the gravitino defined in (85).

这是将作用量从超场约化为分量场的最简方法。此处 $\overline{\Psi}_{a\dot{\alpha}} = e_a{}^m \overline{\Psi}_{m\dot{\alpha}}$ 是 (85) 中定义的引力微子。

In this subsection, we have provided abstract definitions of locally super-conformal action principles. Relevant supergravity-matter models and explicit examples of $\mathcal{L}$ and $\mathcal{L}_c$, respectively, satisfying (74) and (75), will be given in sections ”Matter Multiplets in Conformal Supergravity” and ”Off-Shell Models for Pure Supergravity.”

在本小节中, 我们给出了局域超共形作用量原理的抽象定义。分别满足 (74) 和 (75) 的相关超引力-物质模型以及 $\mathcal{L}$ 和 $\mathcal{L}_c$ 的具体实例，将在“共形超引力中的物质多重态”和“脱壳纯超引力模型”两节中给出。

Component Reduction and the Weyl Multiplet

分量约化与 Weyl 多重态

Having formulated conformal superspace in the preceding subsections, it is instructive to utilise it in reproducing the results of section ”Local Superconformal Transformations.” Specifically, we will briefly describe the Weyl multiplet of conformal supergravity (see, e.g., [28, 33] for pedagogical reviews) and Q - supersymmetry transformations of the corresponding fields. As described earlier, the former involves a set of gauge one-forms: the vielbein $e_m{}^a$, gravitino $\Psi_m{}^\alpha$, $\mathrm{U}(1)_R$ gauge field $\mathfrak{A}_m$, and dilatation gauge field b_m . Modulo purely gauge degrees of freedom, they may be shown to be the only independent geometric fields and arise as the lowest components of the superforms

在前述小节构造共形超空间后，利用它重现“局部超共形变换”一节的结果颇具启发性。我们将简要介绍共形超引力的 Weyl 多重态 (教学综述参见例如 [28, 33]) 以及对应场的 Q 超对称变换。如前文所述，Weyl 多重态包含一组规范 1-形式: 标架场 $e_m{}^a$ 、引力微子 $\Psi_m{}^\alpha$, $\mathrm{U}(1)_R$ 、规范场 $\mathfrak{A}_m$ 和伸缩规范场 b_m 。除去纯规范自由度，可证明它们是仅有的独立几何场，且作为超形式的最低分量出现

$$e_m{}^a := E_m{}^a|,\ \Psi_m{}^\alpha := 2E_m{}^\alpha|,\ \mathfrak{A}_m := \Phi_m|,\ b_m := B_m|, \tag{85}$$

where the bar projection of a superfield $\Xi\left(x, \theta, \bar{\theta}\right)$ is defined by $\Xi\,| := \Xi|_{\theta=\bar{\theta}=0}$.

其中超场 $\Xi\left(x,\theta,\bar{\theta}\right)$ 的横杠投影由 $\Xi\,| := \Xi|_{\theta=\bar{\theta}=0}$ 定义。

It remains to compute the Q -supersymmetry transformations of the fields (85) and show that they do indeed form the Weyl multiplet. By employing (54), their transformations when $\mathcal{K}(\xi) = \xi^\alpha\nabla_\alpha + \bar{\xi}_{\dot{\alpha}}\bar{\nabla}^{\dot{\alpha}}$ may be shown to be:

仍需计算 (85) 中各场的 Q 超对称变换，并证明它们确实构成 Weyl 多重态。利用 (54) 可证明，当 $\mathcal{K}(\xi) = \xi^\alpha\nabla_\alpha + \bar{\xi}_{\dot{\alpha}}\bar{\nabla}^{\dot{\alpha}}$ 时，它们的变换为:

$$\delta_{\mathcal{K}}(\varepsilon)\, e_m{}^a = \mathrm{i}\left(\varepsilon\sigma^a\bar{\Psi}_m - \Psi_m\sigma^a\bar{\varepsilon}\right), \tag{86a}$$

$$\delta_{\mathcal{K}(\varepsilon)}\Psi_m{}^\alpha = 2\hat{\nabla}_m\varepsilon^\alpha, \tag{86b}$$

$$\delta_{\mathcal{K}(\varepsilon)}\mathfrak{A}_m = 3\mathrm{i}\left(\mathfrak{F}_m^\alpha\,|\varepsilon_\alpha - \bar{\mathfrak{F}}_{m\dot{\alpha}}|\,\bar{\varepsilon}^{\dot{\alpha}}\right), \tag{86c}$$

$$\delta_{\mathcal{K}}(\varepsilon)\, b_m = 2\left(\mathfrak{F}_m{}^\alpha\,|\varepsilon_\alpha + \bar{\mathfrak{F}}_{m\dot{\alpha}}|\,\bar{\varepsilon}^{\dot{\alpha}}\right), \tag{86d}$$

where we have denoted $\varepsilon^\alpha := \xi^\alpha\,|$ and $\hat{\nabla}_m$ was defined in (45). Additionally, by a routine analysis of the spinor torsion two-form $\mathcal{T}_{mn}{}^\alpha\,|$, it may be shown that

其中我们记为 $\varepsilon^\alpha := \xi^\alpha\,|$，且 $\hat{\nabla}_m$ 已在 (45) 中定义。此外，对旋量挠率 2-形式 $\mathcal{T}_{mn}{}^\alpha\,|$ 做常规分析可证明:

$$(\sigma^{mn})^{\beta\gamma}\hat{\nabla}_m\Psi_n{}^\alpha + 2\mathrm{i}\varepsilon^{\alpha(\beta}(\sigma^m)^{\gamma)}{}_{\dot{\alpha}}\bar{\mathfrak{F}}_m{}^{\dot{\alpha}}\,| = W^{\alpha\beta\gamma}|, \tag{87a}$$

which implies:

由此可得:

$$W_{\alpha\beta\gamma}\,| = (\sigma^{mn})_{(\alpha\beta}\,\hat{\nabla}_m\Psi_{n\gamma)}, \tag{87b}$$

$$\mathfrak{F}_m{}^\alpha\,| = \frac{\mathrm{i}}{3}\hat{\nabla}_{[m}\bar{\psi}_{n]\dot{\alpha}}(\tilde{\sigma}^n)^{\dot{\alpha}\alpha} - \frac{1}{12}g_{mn}\varepsilon^{nijk}\hat{\nabla}_i\bar{\psi}_{j\dot{\alpha}}(\tilde{\sigma}_k)^{\dot{\alpha}\alpha}. \tag{87c}$$

Then, upon inserting (87c) into (86), the transformations coincide with (44).

随后，将 (87c) 代入 (86)，所得变换与 (44) 一致。

To conclude, it should be noted that the dilatation gauge field b_m describes purely gauge degrees of freedom. This may be seen by noting that, according to (54), it transforms algebraically when $\mathcal{K}(\Lambda) = \Lambda_a K^a$

最后需要注意，伸缩规范场 b_m 仅描述纯规范自由度。根据 (54)，当 $\mathcal{K}(\Lambda) = \Lambda_a K^a$ 时它按代数方式变换，由此即可看出这一点。

$$\delta_{\mathcal{K}(\Lambda)} b_m = -2\Lambda_m \mid . \tag{88}$$

Hence, we impose the gauge $b_m = 0$ by fixing the special conformal gauge freedom. It should also be noted that the remaining fields appearing in (85) are inert under such transformations. Additionally, in order to preserve the gauge $b_m = 0$, each Q -supersymmetry transformation (86) must be accompanied with a compensating special conformal transformation (88) with $\Lambda_m(\varepsilon) = \mathfrak{F}_m{}^{\alpha}|\varepsilon_\alpha + \mathfrak{F}_{m\dot{\alpha}}|\bar{\varepsilon}^{\dot{\alpha}}$. As a result, we have shown that the fields $\left\{e_m{}^a, \Psi_{m\alpha}, \overline{\Psi}_m^{\dot{\alpha}}, \mathfrak{A}_m\right\}$ do indeed constitute the reduced Weyl multiplet introduced in section ”Local Superconformal Transformations.”

因此，我们通过固定特殊共形规范自由度选取规范 $b_m = 0$ 。还需注意，(85) 中其余场在这类变换下是惰性的。此外，为了保持规范 $b_m = 0$ ，每个 Q 超对称变换 (86) 都必须伴随一个参数为 $\Lambda_m(\varepsilon) = \mathfrak{F}_m{}^{\alpha}|\varepsilon_\alpha + \mathfrak{F}_{m\dot{\alpha}}|\bar{\varepsilon}^{\dot{\alpha}}$ 的补偿特殊共形变换 (88)。最终我们证明了，场集合 $\left\{e_m{}^a, \Psi_{m\alpha}, \overline{\Psi}_m^{\dot{\alpha}}, \mathfrak{A}_m\right\}$ 确实构成“局部超共形变换”一节中引入的约化 Weyl 多重态。

Modulo purely gauge degrees of freedom, the Weyl multiplet describes eight bosonic and eight fermionic off-shell fields.

除去纯规范自由度，Weyl 多重态描述 8 个玻色子离壳场与 8 个费米子离壳场。

Other Superspace Formulations for Conformal Supergravity

共形超引力的其他超空间表述

As pointed out in section ”Introduction,” conformal superspace is not the only superspace setting to describe conformal supergravity. The other most popular formulations are as follows: (i) $U(1)$ superspace [31] and (ii) the GWZ superspace [29]. They differ by their structure groups, which are $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{U}(1)_R$ and $\mathrm{SL}(2,\mathbb{C})$, respectively. Both of them can be derived from conformal superspace. Below we describe the relevant degauging procedures.

正如“引言”一节所指出的，共形超空间并非描述共形超引力的唯一超空间框架。其他最常用的表述如下:(i) $U(1)$ 超空间 [31]，以及 (ii)GWZ 超空间 [29]。二者的结构群不同，分别为 $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{U}(1)_R$ 和 $\mathrm{SL}(2,\mathbb{C})$ 。它们都可以从共形超空间推导得出。下文我们将介绍相关的退规范过程。

The U(1) Superspace Geometry

U(1) 超空间几何

According to (52) and (54), under an infinitesimal special superconformal gauge transformation $\mathcal{K} = \Lambda_B K^B$, the dilatation connection transforms as follows:

根据式 (52) 和 (54)，在无穷小特殊超共形规范变换 $\mathcal{K} = \Lambda_B K^B$ 下，伸缩联络的变换形式如下:

$$\delta_{\mathcal{K}} B_A = -2\Lambda_A \tag{89}$$

Thus, it is possible to choose a gauge condition $B_A = 0$, which completely fixes the special superconformal gauge freedom (There is a class of residual gauge transformations preserving the gauge $B_A = 0$. These generate the super-Weyl transformations of $U(1)$ superspace (see the next subsection).). As a result, the corresponding connection is no longer required for the covariance of ∇_A under the residual gauge freedom and may be extracted from ∇_A :

因此，我们可以选取规范条件 $B_A = 0$ ，该条件可以完全固定特殊超共形规范自由度 (存在一类保持规范 $B_A = 0$ 的残余规范变换，它们生成 $U(1)$ 超空间的超外尔变换，参见下一小节)。由此，在残余规范自由度下，∇_A 的协变性不再需要对应联络，可将其从 ∇_A 中剥离出来:

$$\nabla_A = \mathfrak{D}_A - \mathfrak{F}_{AB} K^B. \tag{90}$$

Here the operator $\mathfrak{D}_A$ involves only the Lorentz and $\mathrm{U}(1)_R$ connections.

此处算子 $\mathfrak{D}_A$ 仅包含洛伦兹联络与 $\mathrm{U}(1)_R$ 联络。

The next step is to relate the special superconformal connection $\mathfrak{F}_{AB}$ to the torsion tensor of $U(1)$ superspace. To do this, it is necessary to make use of the relation

下一步是将特殊超共形联络 $\mathfrak{F}_{AB}$ 与 $U(1)$ 超空间的挠率张量联系起来。为此需要用到下述关系

$$[\nabla_A, \nabla_B\} = [\mathfrak{D}_A, \mathfrak{D}_B\} - \left(\mathfrak{D}_A \mathfrak{F}_{BC} - (-1)^{AB} \mathfrak{D}_B \mathfrak{F}_{AC}\right) K^C - \mathfrak{F}_{AC} \left[K^C, \nabla_B\right\}$$

$$+(-1)^{AB} \mathfrak{F}_{BC} \left[K^C, \nabla_A\right\} + (-1)^{BC} \mathfrak{F}_{AC} \mathfrak{F}_{BD} \left[K^D, K^C\right\}. \tag{91}$$

In conjunction with (65), this relation leads to a set of consistency conditions that are equivalent to the Bianchi identities of $U(1)$ superspace [31]. Their solution expresses the components of $\mathfrak{F}_{AB}$ in terms of the torsion tensor of $U(1)$ superspace. We will not provide a detailed analysis for this step and instead refer the reader to the proof in [32]. The outcome of the analysis is as follows:

结合式 (65)，该关系给出了一组相容性条件，它们等价于 $U(1)$ 超空间的比安基恒等式 [31]。其解用 $U(1)$ 超空间的挠率张量表示出了 $\mathfrak{F}_{AB}$ 的分量。本文不对这一步做详细分析，读者可参阅文献 [32] 中的证明。分析结果如下:

$$\mathfrak{F}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta}\bar{R},\ \overline{\mathfrak{F}}_{\dot\alpha\dot\beta} = -\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}R,\ \mathfrak{F}_{\alpha\dot\beta} = -\overline{\mathfrak{F}}_{\dot\beta\alpha} = \frac{1}{4}G_{\alpha\dot\beta}, \tag{92a}$$

$$\mathfrak{F}_{\alpha,\beta\dot\beta} = -\frac{\mathrm{i}}{4}\mathfrak{D}_\alpha G_{\beta\dot\beta} - \frac{\mathrm{i}}{6}\varepsilon_{\alpha\beta}\bar{X}_{\dot\beta} = \mathfrak{F}_{\beta\dot\beta,\alpha}, \tag{92b}$$

$$\overline{\mathfrak{F}}_{\dot\alpha,\beta\dot\beta} = \frac{\mathrm{i}}{4}\overline{\mathfrak{D}}_{\dot\alpha}G_{\beta\dot\beta} + \frac{\mathrm{i}}{6}\varepsilon_{\dot\alpha\dot\beta}X_\beta = \mathfrak{F}_{\beta\dot\beta,\alpha}, \tag{92c}$$

$$\mathfrak{F}_{\alpha\dot\alpha,\beta\dot\beta} = -\frac{1}{8}\left[\mathfrak{D}_\alpha, \overline{\mathfrak{D}}_{\dot\alpha}\right] G_{\beta\dot\beta} - \frac{1}{12}\varepsilon_{\dot\alpha\dot\beta}\mathfrak{D}_\alpha X_\beta + \frac{1}{12}\varepsilon_{\alpha\beta}\overline{\mathfrak{D}}_{\dot\alpha}\bar{X}_{\dot\beta}$$

$$+\frac{1}{2}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{R}R+\frac{1}{8}G_{\alpha\dot{\beta}}G_{\beta\dot{\alpha}} \tag{92d}$$

where R and X_α are complex chiral:

其中 R 和 X_α 为复手征场:

$$\bar{\mathfrak{D}}_{\dot{\alpha}}R=0,\ \mathbb{Y}R=-2R, \tag{93a}$$

$$\bar{\mathfrak{D}}_{\dot{\alpha}}X_\alpha=0,\ \mathbb{Y}X_\alpha=-X_\alpha, \tag{93b}$$

while $G_{\alpha\dot{\alpha}}$ is a real vector superfield. These are related via

而 $G_{\alpha\dot{\alpha}}$ 是实向量超场，它们满足关系

$$X_\alpha=\mathfrak{D}_\alpha R-\bar{\mathfrak{D}}^{\dot{\alpha}}G_{\alpha\dot{\alpha}}. \tag{93c}$$

We now pause and comment on the geometry described by $\mathfrak{D}_A$. In particular, by employing (91), one arrives at the following anti-commutation relation:

现在我们暂停，对 $\mathfrak{D}_A$ 描述的几何做一说明。特别地，利用式 (91) 可以得到如下对易关系:

$$\left\{\mathfrak{D}_\alpha,\bar{\mathfrak{D}}_{\dot{\alpha}}\right\}=-2\mathrm{i}\mathfrak{D}_{\alpha\dot{\alpha}}-G^\beta{}_{\dot{\alpha}}M_{\alpha\beta}+G_\alpha{}^{\dot{\beta}}\bar{M}_{\dot{\alpha}\dot{\beta}}+\frac{3}{2}G_{\alpha\dot{\alpha}}\mathbb{Y}. \tag{94}$$

With the goal of simplicity in performing calculations within U (1) superspace, we prefer to work with a geometry, where the right-hand side of (94) contains no curvature-dependent terms. To this end, we perform the following redefinition:

为了简化在 U (1) 超空间内的计算，我们倾向于采用让式 (94) 右侧不含曲率相关项的几何。为此我们做如下重定义:

$$\mathfrak{D}_{\alpha\dot{\alpha}}=\mathcal{D}_{\alpha\dot{\alpha}}+\frac{\mathrm{i}}{2}G^\beta{}_{\dot{\alpha}}M_{\alpha\beta}-\frac{\mathrm{i}}{2}G_\alpha{}^{\dot{\beta}}\bar{M}_{\dot{\alpha}\dot{\beta}}-\frac{3\mathrm{i}}{4}G_{\alpha\dot{\alpha}}\mathbb{Y}, \tag{95a}$$

$$\mathfrak{D}_\alpha=\mathcal{D}_\alpha,\ \bar{\mathfrak{D}}^{\dot{\alpha}}=\bar{\mathcal{D}}^{\dot{\alpha}}, \tag{95b}$$

where $\mathcal{D}_A$ takes the form

其中 $\mathcal{D}_A$ 形式为

$$\mathcal{D}_A=E_A-\frac{1}{2}\widehat{\Omega}_A{}^{bc}M_{bc}-\mathrm{i}\widehat{\Phi}_A\mathbb{Y}$$

$$=E_A-\widehat{\Omega}_A{}^{\beta\gamma}M_{\beta\gamma}-\widehat{\bar{\Omega}}_A{}^{\dot{\beta}\dot{\gamma}}\bar{M}_{\dot{\beta}\dot{\gamma}}-\mathrm{i}\widehat{\Phi}_A\mathbb{Y}. \tag{96}$$

Here we have attached a hat to each connection superfield to distinguish them from their cousins residing in the conformal covariant derivative ∇_A . In what follows, these hats will be omitted.

此处我们给每个联络超场加上 hat 符号，以区分共形协变导数 ∇_A 中的对应联络超场。在下文中我们将省略该 hat 符号。

Now it may be shown that the algebra obeyed by $\mathcal{D}_A$ takes the form

现在可以证明，$\mathcal{D}_A$ 满足的代数形式为

$$\left\{\mathcal{D}_\alpha, \mathcal{D}_\beta\right\} = -4\bar{R}M_{\alpha\beta}, \left\{\overline{\mathcal{D}}_{\dot\alpha}, \overline{\mathcal{D}}_{\dot\beta}\right\} = 4R\bar{M}_{\dot\alpha\dot\beta}, \tag{97a}$$

$$\left\{\mathcal{D}_\alpha, \overline{\mathcal{D}}_{\dot\alpha}\right\} = -2\mathrm{i}\mathcal{D}_{\alpha\dot\alpha}, \tag{97b}$$

$$\begin{aligned}\left[\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot\beta}\right] = {} & \mathrm{i}\varepsilon_{\alpha\beta}\left(\bar{R}\overline{\mathcal{D}}_{\dot\beta} + G^\gamma{}_{\dot\beta}\mathcal{D}_\gamma - \left(\mathcal{D}^\gamma G^\delta{}_{\dot\beta}\right)M_{\gamma\delta} + 2\bar{W}_{\dot\beta}{}^{\dot\gamma\dot\delta}\bar{M}_{\dot\gamma\dot\delta}\right) \\ & +\mathrm{i}\left(\overline{\mathcal{D}}_{\dot\beta}\bar{R}\right)M_{\alpha\beta} - \frac{\mathrm{i}}{3}\varepsilon_{\alpha\beta}\bar{X}^{\dot\gamma}\bar{M}_{\dot\gamma\dot\beta} - \frac{\mathrm{i}}{2}\varepsilon_{\alpha\beta}\bar{X}_{\dot\beta}\mathbb{Y}\end{aligned} \tag{97c}$$

$$\begin{aligned}\left[\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{D}_{\beta\dot\beta}\right] = {} & -\mathrm{i}\varepsilon_{\dot\alpha\dot\beta}\left(R\mathcal{D}_\beta + G_\beta{}^{\dot\gamma}\overline{\mathcal{D}}_{\dot\gamma} - \left(\overline{\mathcal{D}}^{\dot\gamma} G_\beta{}^{\dot\delta}\right)\bar{M}_{\dot\gamma\dot\delta} + 2W_\beta{}^{\gamma\delta}M_{\gamma\delta}\right) \\ & -\mathrm{i}\left(\mathcal{D}_\beta R\right)\bar{M}_{\dot\alpha\dot\beta} + \frac{\mathrm{i}}{3}\varepsilon_{\dot\alpha\dot\beta}X^\gamma M_{\gamma\beta} - \frac{\mathrm{i}}{2}\varepsilon_{\dot\alpha\dot\beta}X_\beta\mathbb{Y}\end{aligned} \tag{97d}$$

which lead to

由此可得

$$\left[\mathcal{D}_{\alpha\dot\alpha}, \mathcal{D}_{\beta\dot\beta}\right] = \varepsilon_{\alpha\beta}\overline{\chi}_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}\chi_{\alpha\beta}, \tag{97e}$$

$$\begin{aligned}\chi_{\alpha\beta} = {} & -\mathrm{i}G_{(\alpha}{}^{\dot\gamma}\mathcal{D}_{\beta)\dot\gamma} + \frac{1}{2}\mathcal{D}_{(\alpha}R\mathcal{D}_{\beta)} + \frac{1}{2}\mathcal{D}_{(\alpha}G_{\beta)}{}^{\dot\gamma}\overline{\mathcal{D}}_{\dot\gamma} + W_{\alpha\beta}{}^\gamma\mathcal{D}_\gamma \\ & +\frac{1}{6}X_{(\alpha}\mathcal{D}_{\beta)} + \frac{1}{4}\left(\mathcal{D}^2 - 8R\right)\bar{R}M_{\alpha\beta} + \mathcal{D}_{(\alpha}W_{\beta)}{}^{\gamma\delta}M_{\gamma\delta} \\ & -\frac{1}{6}\mathcal{D}_{(\alpha}X^\gamma M_{\beta)\gamma} - \frac{1}{2}\mathcal{D}_{(\alpha}\overline{\mathcal{D}}^{\dot\gamma} G_{\beta)}{}^{\dot\delta}\bar{M}_{\dot\gamma\dot\delta} + \frac{1}{4}\mathcal{D}_{(\alpha}X_{\beta)}\mathbb{Y}\end{aligned} \tag{97f}$$

$$\begin{aligned}\overline{\chi}_{\dot\alpha\dot\beta} = {} & \mathrm{i}G^\gamma\left(\dot\alpha\mathcal{D}_{\gamma\dot\beta}\right) - \frac{1}{2}\overline{\mathcal{D}}_{(\dot\alpha}\bar{R}\overline{\mathcal{D}}_{\dot\beta)} - \frac{1}{2}\overline{\mathcal{D}}_{(\dot\alpha}G^\gamma{}_{\dot\beta)}\mathcal{D}_\gamma - \bar{W}_{\dot\alpha\dot\beta}{}^{\dot\gamma}\overline{\mathcal{D}}_{\dot\gamma} \\ & -\frac{1}{6}\bar{X}_{(\dot\alpha}\overline{\mathcal{D}}_{\dot\beta)} + \frac{1}{4}\left(\overline{\mathcal{D}}^2 - 8\bar{R}\right)R\bar{M}_{\dot\alpha\dot\beta} - \overline{\mathcal{D}}_{(\dot\alpha}\bar{W}_{\dot\beta)}{}^{\dot\gamma\dot\delta}\bar{M}_{\dot\gamma\dot\delta} \\ & +\frac{1}{6}\overline{\mathcal{D}}_{(\dot\alpha}\bar{X}^{\dot\gamma}\bar{M}_{\dot\beta)\dot\gamma} + \frac{1}{2}\overline{\mathcal{D}}_{(\dot\alpha}\mathcal{D}^\gamma G^\delta{}_{\dot\beta)}M_{\gamma\delta} + \frac{1}{4}\overline{\mathcal{D}}_{(\dot\alpha}\bar{X}_{\dot\beta)}\mathbb{Y}.\end{aligned} \tag{97g}$$

These relations should be supplemented with the following Bianchi identities:

这些关系需要补充下述比安基恒等式:

$$\mathcal{D}^{\alpha}X_{\alpha}=\bar{\mathcal{D}}_{\dot{\alpha}}\bar{X}^{\dot{\alpha}} \tag{98a}$$

$$\bar{\mathcal{D}}_{\dot{\alpha}}W_{\alpha\beta\gamma}=0, \tag{98b}$$

$$\mathcal{D}^{\gamma}W_{\alpha\beta\gamma}=\mathrm{i}\mathcal{D}_{(\alpha}{}^{\dot{\gamma}}G_{\beta)\dot{\gamma}}-\frac{1}{3}\mathcal{D}_{(\alpha}X_{\beta)}. \tag{98c}$$

In particular, it should be noted that (98a) implies X_{α} is the chiral field strength of a $U(1)$ vector multiplet. The geometry described above is the $U(1)$ superspace geometry [13, 31] in the form described in [45, 46] .

特别需要注意的是，式 (98a) 表明 X_{α} 是 $U(1)$ 向量多重态的手征场强。上述几何就是 [45, 46] 中给出形式的 [13, 31]，即 $U(1)$ 超空间几何。

To conclude our discussion of the U (1) superspace geometry, we make two comments. Firstly, one may check that degauging the relation (68) gives

在结束我们对 U (1) 超空间几何的讨论之前，我们给出两点说明。第一，可以验证对式 (68) 退去规范后可得

$$\phi_{\alpha(n)}=-\frac{1}{4}\left(\bar{\mathcal{D}}^{2}-4R\right)\psi_{\alpha(n)},\ \bar{\mathcal{D}}_{\dot{\beta}}\phi_{\alpha(n)}=0. \tag{99}$$

Secondly, integration by parts is remarkably simple in U (1) superspace:

第二，分部积分在 U (1) 超空间中非常简便:

$$\int \mathrm{d}^{4|4}zE(-1)^{\varepsilon_{A}}\mathcal{D}_{A}\mathcal{V}^{A}=0, \tag{100}$$

where $\mathcal{V}^{A}$ is arbitrary (In conformal superspace, integration by parts requires special care [32].).

其中 $\mathcal{V}^{A}$ 是任意的 (在共形超空间中，分部积分需要特别注意 [32]。)

The Super-Weyl Transformations of U(1) Superspace

U(1) 超空间的超 Weyl 变换

In the previous subsection, we made use of the special conformal gauge freedom to degauge from conformal to U (1) superspace. Here we will show that the residual dilatation symmetry manifests in the latter as super-Weyl transformations.

在上一小节中，我们利用特殊共形规范自由度完成了从共形超空间到 U (1) 超空间的退规范操作。本文将说明，剩余伸缩对称性在后者中体现为超 Weyl 变换。

Specifically, to preserve the gauge $B_A = 0$, every local dilatation transformation with parameter $\sum$ should be accompanied by a compensating special conformal one, $\Lambda^B(\sum)$

具体而言，为了保持 $B_A = 0$ 规范，每个参数为 $\sum$ 的局域伸缩变换都必须伴随一个补偿特殊共形变换 $\Lambda^B(\sum)$

$$\mathcal{K}\left(\sum\right) = \Lambda_B\left(\sum\right) K^B + \sum \mathbb{D} \ \Rightarrow\ \delta_{\mathcal{K}(\sum)} B_A = 0. \tag{101}$$

We then arrive at the following constraints:

我们由此得到如下约束条件:

$$\Lambda_A\left(\sum\right) = \frac{1}{2}\nabla_A \sum. \tag{102}$$

As a result, we define the following transformation:

据此，我们定义如下变换:

$$\delta_{\sum}\nabla_A = \delta_{\sum}\mathfrak{D}_A - \delta_{\sum}\left(\mathfrak{F}_{AB}K^B\right) = \left[\mathcal{K}\left(\sum\right), \nabla_A\right]. \tag{103}$$

By making use of (95) and (92), we arrive at the following transformation laws for the $U(1)$ superspace covariant derivatives:

利用式 (95) 和 (92)，我们推导出 $U(1)$ 超空间协变导数满足如下变换规律:

$$\delta_{\sum}\mathcal{D}_\alpha = \frac{1}{2}\sum\mathcal{D}_\alpha + 2\mathcal{D}^\beta\sum M_{\beta\alpha} - \frac{3}{2}\mathcal{D}_\alpha\sum\mathbb{Y}, \tag{104a}$$

$$\delta_{\sum}\overline{\mathcal{D}}_{\dot\alpha} = \frac{1}{2}\sum\overline{\mathcal{D}}_{\dot\alpha} + 2\overline{\mathcal{D}}^{\dot\beta}\sum\bar{M}_{\dot\beta\dot\alpha} + \frac{3}{2}\overline{\mathcal{D}}_{\dot\alpha}\sum\mathbb{Y}, \tag{104b}$$

$$\begin{aligned}\delta_{\sum}\mathcal{D}_{\alpha\dot\alpha} = &\sum\mathcal{D}_{\alpha\dot\alpha} + \mathrm{i}\mathcal{D}_\alpha\sum\overline{\mathcal{D}}_{\dot\alpha} + \mathrm{i}\overline{\mathcal{D}}_{\dot\alpha}\sum\mathcal{D}_\alpha + \mathrm{i}\overline{\mathcal{D}}_{\dot\alpha}\mathcal{D}^\beta\sum M_{\beta\alpha}\\ &+\mathrm{i}\mathcal{D}_\alpha\overline{\mathcal{D}}^{\dot\beta}\sum\bar{M}_{\dot\beta\dot\alpha} + \frac{3}{4}\mathrm{i}\left[\mathcal{D}_\alpha, \overline{\mathcal{D}}_{\dot\alpha}\right]\sum\mathbb{Y},\end{aligned} \tag{104c}$$

while the torsion superfields arising from the degauged torsion $\mathfrak{F}_{AB}$ transform as follows:

而退规范后挠率 $\mathfrak{F}_{AB}$ 得到的挠超场按下述规则变换:

$$\delta_{\sum}R = \sum R + \frac{1}{2}\overline{\mathcal{D}}^2\sum \tag{105a}$$

$$\delta_{\sum}G_{\alpha\dot\alpha} = \sum G_{\alpha\dot\alpha} + \left[\mathcal{D}_\alpha, \overline{\mathcal{D}}_{\dot\alpha}\right]\sum, \tag{105b}$$

$$\delta_{\sum}X_\alpha = \frac{3}{2}\sum X_\alpha - \frac{3}{2}\left(\overline{\mathcal{D}}^2 - 4R\right)\mathcal{D}_\alpha\sum. \tag{105c}$$

Finally, as the super-Weyl tensor is conformally covariant, its transformation law is readily obtained via

最后，由于超 Weyl 张量是共形协变的，可通过如下方式直接得到它的变换规律

$$\delta_{\sum} W_{\alpha\beta\gamma} = \left(\Lambda_B \left(\sum\right) K^B + \sum \mathbb{D}\right) W_{\alpha\beta\gamma} = \frac{3}{2} \sum W_{\alpha\beta\gamma}. \tag{105d}$$

The relations (104) and (105) give the super-Weyl transformations in U (1) superspace [13,31] (see also [45,46]). The conditions (60), which define a primary superfield U , are equivalent to the following:

关系式 (104) 和 (105) 给出了 U (1) 超空间中的超 Weyl 变换 [13,31](另见文献 [45,46])。定义原超场 U 的条件 (60) 等价于下述表述:

$$\delta_{\sum} U = w \sum U, \ \mathbb{Y} U = cU, \tag{106}$$

in $U(1)$ superspace.

在 $U(1)$ 超空间中成立。

The $U(1)$ superspace formulation was fully developed in the book [13], in which various applications were also given.

$U(1)$ 超空间表述在文献 [13] 中得到了完整发展，其中也给出了多类应用。

The Grimm-Wess-Zumino Formulation

格里姆-韦斯-祖米诺表述

As pointed out in Section "The U(1) Superspace Geometry," the covariantly chiral spinor X_α is the field strength of an Abelian vector multiplet. It follows from (105c) that the super-Weyl gauge freedom allows us to choose the gauge

正如在“U(1) 超空间几何”一节中指出的，协变手性旋量 X_α 是阿贝尔矢量多重态的场强。由式 (105c) 可知，超外尔规范自由度允许我们选取规范

$$X_\alpha = 0\,. \tag{107}$$

In this gauge the $\mathrm{U}(1)_R$ curvature vanishes, in accordance with (97), and therefore the $\mathrm{U}(1)_R$ connection may be gauged away:

根据式 (97)，在此规范下 $\mathrm{U}(1)_R$ 曲率为零，因此 $\mathrm{U}(1)_R$ 联络可以被规范掉:

$$\Phi_A = 0\,. \tag{108}$$

Then, the algebra of covariant derivatives (97) reduces to that describing the GWZ geometry [29].

随后，协变导数的代数 (97) 约化为描述 GWZ 几何的代数 [29]。

Equation (105c) tells us that imposing the condition $X_\alpha = 0$ does not fix completely the super-Weyl freedom. The residual transformations are generated by parameters of the form

式 (105c) 表明，加上条件 $X_\alpha = 0$ 并未完全固定超外尔自由度。剩余变换由如下形式的参数生成

$$\sum = \frac{1}{2}\left(\sigma + \overline{\sigma}\right), \ \overline{\mathcal{D}}_{\dot\alpha}\sigma = 0. \tag{109}$$

However, in order to preserve the $\mathrm{U}(1)_R$ gauge $\Phi_A = 0$, every residual super-Weyl transformation (109) must be accompanied by a compensating $\mathrm{U}(1)_R$ transformation with

但为了保留 $\mathrm{U}(1)_R$ 规范 $\Phi_A = 0$ ，每个剩余超外尔变换 (109) 都必须伴随一个补偿 $\mathrm{U}(1)_R$ 变换，满足

$$\rho = \frac{3}{4}\mathrm{i}\left(\sigma - \overline{\sigma}\right). \tag{110}$$

This leads to the transformation [47]

由此得到变换 [47]

$$\delta_\sigma \mathcal{D}_\alpha = \left(\overline{\sigma} - \frac{1}{2}\sigma\right)\mathcal{D}_\alpha + \left(\mathcal{D}^\beta\sigma\right)M_{\alpha\beta}, \tag{111a}$$

$$\delta_\sigma \overline{\mathcal{D}}_{\dot\alpha} = \left(\sigma - \frac{1}{2}\overline{\sigma}\right)\overline{\mathcal{D}}_{\dot\alpha} + \left(\overline{\mathcal{D}}^{\dot\beta}\overline{\sigma}\right)\bar{M}_{\dot\alpha\dot\beta}, \tag{111b}$$

$$\delta_\sigma \mathcal{D}_{\alpha\dot\alpha} = \frac{1}{2}\left(\sigma + \overline{\sigma}\right)\mathcal{D}_{\alpha\dot\alpha} + \frac{\mathrm{i}}{2}\left(\overline{\mathcal{D}}_{\dot\alpha}\overline{\sigma}\right)\mathcal{D}_\alpha + \frac{\mathrm{i}}{2}\left(\mathcal{D}_\alpha\sigma\right)\overline{\mathcal{D}}_{\dot\alpha}$$

$$+\left(\mathcal{D}^\beta{\dot\alpha}\sigma\right)M_{\alpha\beta} + \left(\mathcal{D}_\alpha{}^{\dot\beta}\overline{\sigma}\right)\bar{M}_{\dot\alpha\dot\beta}. \tag{111c}$$

The torsion tensors transform as follows:

挠率张量的变换规则如下:

$$\delta_\sigma R = 2\sigma R + \frac{1}{4}\left(\overline{\mathcal{D}}^2 - 4R\right)\overline{\sigma}, \tag{112a}$$

$$\delta_\sigma G_{\alpha\dot\alpha} = \frac{1}{2}\left(\sigma + \overline{\sigma}\right)G_{\alpha\dot\alpha} + \mathrm{i}\mathcal{D}_{\alpha\dot\alpha}\left(\sigma - \overline{\sigma}\right), \tag{112b}$$

$$\delta_\sigma W_{\alpha\beta\gamma} = \frac{3}{2}\sigma W_{\alpha\beta\gamma}. \tag{112c}$$

The conditions (106), which define a primary superfield U , turn into

定义原初超场 U 的条件 (106) 在 GWZ 方法中变为

$$\delta_\sigma U = \left(p\sigma + q\bar{\sigma}\right) U,\ p+q=w,\ p-q=-\frac{3}{2}c, \tag{113}$$

in the GWZ approach.

在 GWZ 方法中。

Let us fix a background curved superspace $\left(\mathcal{M}^{4|4}, \mathcal{D}\right)$. A supervector field $\xi = \xi^B E_B$ on this superspace is called conformal Killing, if there exists a Lorentz parameter $K^{bc}[\xi]$ and a super-Weyl chiral parameter $\sigma[\xi]$, such that

我们固定一个背景弯曲超空间 $\left(\mathcal{M}^{4|4}, \mathcal{D}\right)$。该超空间上的超矢量场 $\xi = \xi^B E_B$ 称为共形 Killing 超矢量场，如果存在洛伦兹参数 $K^{bc}[\xi]$ 和超外尔手性参数 $\sigma[\xi]$，使得

$$\left[\xi^B \mathcal{D}_B + \frac{1}{2}K^{bc}[\xi] M_{bc}, \mathcal{D}_A\right] + \delta_{\sigma[\xi]}\mathcal{D}_A = 0. \tag{114}$$

In other words, the coordinate transformation generated by ξ is accompanied by certain Lorentz and super-Weyl transformations, such that the superspace geometry does not change. It can be shown [35] that the equation (114) uniquely determines the spinor components of $\xi^B = \left(\xi^b, \xi^\beta, \bar{\xi}_{\dot\beta}\right)$ and the parameters $K^{bc}[\xi]$ and $\sigma[\xi]$ in terms of ξ^b , and the latter obeys the equation

换句话说，由 ξ 生成的坐标变换伴随特定的洛伦兹变换和超外尔变换，使得超空间几何保持不变。可以证明 [35]，方程 (114) 由 ξ^b 唯一确定 $\xi^B = \left(\xi^b, \xi^\beta, \bar{\xi}_{\dot\beta}\right)$ 的旋量分量以及参数 $K^{bc}[\xi]$ 和 $\sigma[\xi]$，且 ξ^b 满足方程

$$\mathcal{D}_{(\alpha}\xi_{\beta)\dot\beta} = 0 \Leftrightarrow \bar{\mathcal{D}}_{(\dot\alpha}\xi_{\beta\dot\beta)} = 0. \tag{115}$$

The set of all conformal Killing supervector fields on $\left(\mathcal{M}^{4|4}, \mathcal{D}\right)$ constitutes the superconformal algebra of $\left(\mathcal{M}^{4|4}, \mathcal{D}\right)$. Given a super-Weyl invariant theory on $\left(\mathcal{M}^{4|4}, \mathcal{D}\right)$ described by primary superfields U , its action is invariant under the superconformal transformations

$\left(\mathcal{M}^{4|4}, \mathcal{D}\right)$ 上所有共形 Killing 超矢量场的集合构成 $\left(\mathcal{M}^{4|4}, \mathcal{D}\right)$ 的超共形代数。对于 $\left(\mathcal{M}^{4|4}, \mathcal{D}\right)$ 上由原初超场 U 描述的超外尔不变理论，其作用量在超共形变换下不变，即

$$\delta_\xi U = \mathcal{K}[\xi] U,\ \mathcal{K}[\xi] = \xi^B \mathcal{D}_B + \frac{1}{2}K^{bc}[\xi] M_{bc} + p\sigma[\xi] + q\bar{\sigma}[\xi] \tag{116}$$

for an arbitrary conformal Killing supervector field ξ . In the case that $\left(\mathcal{M}^{4|4}, \mathcal{D}\right)$ coincides with Minkowski superspace, $\left(\mathbb{M}^{4|4}, D\right)$, the superconformal Killing equation (114) is equivalent to (6) and the transformation law (116) to (22).

对任意共形 Killing 超矢量场 ξ 成立。当 $\left(\mathcal{M}^{4|4}, \mathcal{D}\right)$ 为闵可夫斯基超空间 $\left(\mathbb{M}^{4|4}, D\right)$ 时，超共形 Killing 方程 (114) 等价于式 (6)，变换法则 (116) 等价于式 (22)。

The GWZ formulation has been used in most applications of $\mathcal{N} = 1$ superfield supergravity. It is reviewed in several textbooks (see, e.g., [17,35,48]).

GWZ 表述已被应用于 $\mathcal{N}=1$ 超场超引力的大多数研究中，多本教材都对其做了综述 (例如，参见文献 [17,35,48])。

Supergravity Prepotentials

超引力预势

The constraints on the GWZ geometry [8, 29] were solved by Siegel [18] in terms of unconstrained prepotentials. This solution was extended to non-minimal supergravity ($n \neq -1/3, 0$) , by Gates and Siegel [6], and then to 1 (1) superspace in the book [13]. Here we review the original solution given in [18]. The prepotential description of conformal superspace was worked out in [32], and the interested reader is referred to the original publication.

西格尔利用无约束预势求解了 GWZ 几何的约束条件 [8, 29]。盖茨和西格尔将该解推广到了非极小超引力 ($n \neq -1/3, 0$) ，之后在文献 [13] 中又推广到了 1 (1) 超空间。本文我们综述文献 [18] 中给出的原始解。共形超空间的预势描述已在文献 [32] 中完成，感兴趣的读者可查阅原文。

The covariant derivatives have the form

协变导数具有如下形式

$$\mathcal{D}_A = E_A - \frac{1}{2}\Omega_A{}^{bc}M_{bc} \tag{117}$$

and obey the graded commutation relations

且满足阶化对易关系

$$[\mathcal{D}_A, \mathcal{D}_B\} = -\mathcal{T}_{AB}{}^C\mathcal{D}_C - \frac{1}{2}\mathcal{R}_{AB}{}^{cd}M_{cd}, \tag{118}$$

where the torsion and curvature tensors are read off from (97) by setting $X_\alpha = 0$.

其中挠率张量和曲率张量可通过令 $X_\alpha = 0$ 从式 (97) 中读出。

The gauge group of conformal supergravity is generated by the general coordinate $\left(K^N\right)$, local Lorentz $\left(K^{bc}\right)$, and super-Weyl (σ and $\bar{\sigma}$) transformations. The combined general coordinate and Local Lorentz transformation acts on $\mathcal{D}_A$ and a tensor superfield $\mathcal{T}$ (with suppressed indices) by the rule

共形超引力的规范群由广义坐标 $\left(K^N\right)$ 、局域洛伦兹 $\left(K^{bc}\right)$ 、超外尔 (σ 和 $\bar{\sigma}$) 变换生成。组合后的广义坐标与局域洛伦兹变换按照下述规则作用于 $\mathcal{D}_A$ 和张量超场 $\mathcal{T}$ (隐去指标)

$$\mathcal{D}'_A = \mathrm{e}^{\mathcal{K}}\mathcal{D}_A\mathrm{e}^{-\mathcal{K}}, \ \mathcal{T}' = \mathrm{e}^{\mathcal{K}}\mathcal{T}, \ \mathcal{K} = K^N\partial_N + \frac{1}{2}K^{bc}M_{bc}. \tag{119}$$

The super-Weyl transformation of $\mathcal{D}_A$ is given by Eq. (111), with the parameter σ being covariantly chiral. Given a primary tensor superfield U , its super-Weyl transformation law is given by Eq. (113).

$\mathcal{D}_A$ 的超外尔变换由式 (111) 给出，其中参数 σ 是协变手征的。给定原初张量超场 U，其超外尔变换规则由式 (113) 给出。

Spinor Covariant Derivatives

旋量协变导数

Nontrivial information is contained in the relations (97a) and (97b). First of all, the spinor components of the connection $\Omega_A{}^{bc} = \left(\Omega_a{}^{bc}, \Omega_\alpha{}^{bc}, \overline{\Omega}^{\dot\alpha bc}\right)$ are determined in terms of the anholonomy coefficients, $C_{AB}{}^C$, defined by

关系式 (97a) 和 (97b) 中包含非平凡信息。首先，联络 $\Omega_A{}^{bc} = \left(\Omega_a{}^{bc}, \Omega_\alpha{}^{bc}, \overline{\Omega}^{\dot\alpha bc}\right)$ 的旋量分量由非完整系数 $C_{AB}{}^C$ 确定，其定义为

$$[E_A, E_B\} = C_{AB}{}^C E_C. \tag{120}$$

In particular, for $\frac{1}{2}\Omega_\alpha{}^{bc}M_{bc} = \Omega_\alpha{}^{\beta\gamma}M_{\beta\gamma} + \Omega_\alpha{}^{\dot\beta\dot\gamma}\bar{M}_{\dot\beta\dot\gamma}$, we obtain

特别地，对 $\frac{1}{2}\Omega_\alpha{}^{bc}M_{bc} = \Omega_\alpha{}^{\beta\gamma}M_{\beta\gamma} + \Omega_\alpha{}^{\dot\beta\dot\gamma}\bar{M}_{\dot\beta\dot\gamma}$ ，我们得到

$$\Omega_{\alpha\beta\gamma} = \frac{1}{2}\left(C_{\alpha\beta\gamma} + C_{\alpha\gamma\beta} - C_{\beta\gamma\alpha}\right),\ \Omega_{\alpha\dot\beta\dot\gamma} = -C_{\alpha\dot\beta\dot\gamma}. \tag{121}$$

Secondly, since the curvature $R_{\alpha\beta}{}^{\dot\beta\dot\gamma}\bar{M}_{\dot\beta\dot\gamma}$ vanishes, the connection $\Omega_\alpha{}^{\dot\beta\dot\gamma}\bar{M}_{\dot\beta\dot\gamma}$ is flat,

其次，由于曲率 $R_{\alpha\beta}{}^{\dot\beta\dot\gamma}\bar{M}_{\dot\beta\dot\gamma}$ 为零，联络 $\Omega_\alpha{}^{\dot\beta\dot\gamma}\bar{M}_{\dot\beta\dot\gamma}$ 是平坦的，

$$\Omega_\alpha{}^{\dot\beta\dot\gamma}\bar{M}_{\dot\beta\dot\gamma} = -\bar{g}^{-1}E_\alpha\bar{g},\ \bar{g} = \exp\left(\bar{L}^{\dot\beta\dot\gamma}\bar{M}_{\dot\beta\dot\gamma}\right), \tag{122a}$$

$$\overline{\Omega}_{\dot\alpha}{}^{\beta\gamma}M_{\beta\gamma} = -g^{-1}\bar{E}_{\dot\alpha}g,\ g = \exp\left(L^{\beta\gamma}M_{\beta\gamma}\right). \tag{122b}$$

It follows from (97a) that the spinor components E_α of the inverse supervielbein E_A form a closed algebra in the sense that $\{E_\alpha, E_\beta\} = C_{\alpha\beta}{}^\gamma E_\gamma$. Then, the Frobenius theorem implies that E_α is a linear combination of coordinate supervector fields:

由 (97a) 可知，逆超标架 E_A 的旋量分量 E_α 满足 $\{E_\alpha, E_\beta\} = C_{\alpha\beta}{}^\gamma E_\gamma$ 意义下的闭合代数。根据弗罗贝尼乌斯定理，E_α 可表示为坐标超向量场的线性组合:

$$E_\alpha = FN_\alpha^\mu\widehat{E}_\mu,\ \widehat{E}_\mu = \mathrm{e}^W\partial_\mu\mathrm{e}^{-W},\ W = W^N\partial_N, \tag{123a}$$

$$\bar{E}_{\dot\alpha} = \bar{F}\bar{N}_{\dot\alpha}{}^{\dot\mu}\widehat{\bar{E}}_{\dot\mu},\ \widehat{\bar{E}}_{\dot\mu} = -\mathrm{e}^{\bar{W}}\bar\partial_{\dot\mu}\mathrm{e}^{-\bar{W}},\ \bar{W} = \bar{W}^N\partial_N. \tag{123b}$$

Here the matrix $N = (N_\alpha{}^\mu)$ is unimodular, $N \in \mathrm{SL}(2,\mathbb{C})$, and the scalar F is nowhere vanishing. The complex supervector field W^N is unconstrained.

此处矩阵 $N = (N_\alpha{}^\mu)$ 是幺模矩阵，即 $N \in \mathrm{SL}(2,\mathbb{C})$，标量 F 处处非零，复超向量场 W^N 无约束。

Consider a covariantly chiral tensor superfield $\Psi_{\alpha_1 \dots \alpha_n}$. Making use of (122) and (123) gives

考虑协变手征张量超场 $\Psi_{\alpha_1 \dots \alpha_n}$，利用 (122) 和 (123) 可得

$$\bar{\mathcal{D}}_{\dot\beta} \Psi_{\alpha_1 \dots \alpha_n} = 0 \Leftrightarrow \Psi_{\alpha_1 \dots \alpha_n}\left(x, \theta, \bar\theta\right) = g^{-1} \mathrm{e}^{\bar{W}} \hat\Psi_{\alpha_1 \dots \alpha_n}(x, \theta), \tag{124}$$

where g is given by (122b)

其中 g 由 (122b) 给出

Local Lorentz transformations correspond to setting $K^N = 0$ in (119). They act only on the matrix N in (123a),

局域洛伦兹变换对应于将 (119) 中的 $K^N = 0$ 取为该值，它仅作用于 (123a) 中的矩阵 N，

$$N' = \exp\left(\frac{1}{2} K^{ab} \sigma_{ab}\right) N \tag{125}$$

and therefore, it is possible to choose

因此，我们可以选取

$$N = \mathbb{1}\,. \tag{126}$$

This gauge condition is useful for several applications (see below). Another useful gauge fixing of the local Lorentz symmetry is

该规范条件在多种应用中都很有用 (见下文)。局域洛伦兹对称性的另一种常用规范固定条件是

$$\Omega_\alpha{}^{\dot\beta\dot\gamma} = 0. \tag{127}$$

The Λ Gauge Group

Λ 规范群

General coordinate transformations correspond to setting $K^{bc} = 0$ in (119). They act on the building blocks in (123a) as follows:

广义坐标变换对应式 (119) 中设定 $K^{bc} = 0$，它对式 (123a) 中构造块的作用如下:

$$F' = e^{K}F,\ N' = e^{K}N,\ e^{W'} = e^{K}e^{W}. \tag{128}$$

Once the constraints on the torsion have been partially solved in terms of F, N and the complex unconstrained prepotential W^{N} , there may appear an additional gauge freedom. To uncover it, consider a covariantly chiral scalar superfield Φ

利用 F, N 和复无约束预势 W^{N} 部分求解挠率约束后，可能会出现额外的规范自由度。为揭示这一点，考虑协变手征标量超场 Φ

$$\overline{\mathcal{D}}_{\dot{\alpha}}\Phi = 0 \Leftrightarrow \Phi\left(x, \theta, \bar{\theta}\right) = e^{\bar{W}}\hat{\Phi}(x, \theta),\ \bar{\partial}_{\dot{\mu}}\hat{\Phi} = 0. \tag{129}$$

Its transformation law under (119) is

它在 (119) 下的变换规律为

$$\Phi' = e^{K}\Phi \Leftrightarrow e^{\bar{W}'} = e^{K}e^{\bar{W}},\ \hat{\Phi}' = \hat{\Phi}. \tag{130}$$

We can introduce a new gauge transformation defined by

我们可以引入一个由下式定义的新规范变换:

$$e^{\bar{W}'} = e^{\bar{W}}e^{-\Lambda},\ \hat{\Phi}' = e^{\Lambda}\hat{\Phi} = \exp\left(\lambda^{n}\partial_{n} + \lambda^{\upsilon}\partial_{\upsilon}\right)\hat{\Phi}, \tag{131a}$$

$$\Lambda = \lambda^{N}\partial_{N} = \lambda^{n}\partial_{n} + \lambda^{\upsilon}\partial_{\upsilon} + \lambda_{\dot{\upsilon}}\bar{\partial}^{\dot{\upsilon}},\ \bar{\partial}_{\dot{\mu}}\lambda^{n} = 0,\ \bar{\partial}_{\dot{\mu}}\lambda^{\upsilon} = 0, \tag{131b}$$

which does not change Φ and which preserves the chirality of $\hat{\Phi}$. Of special significance is the fact that the spinor parameter $\lambda_{\dot{\upsilon}}$ is unconstrained. It is obvious that the gauge transformations (131) form a group, which is known as the Λ gauge group. It turns out that the Λ -transformation of W ,

它不改变 Φ ，且保持 $\hat{\Phi}$ 的手征性。特别重要的一点是，旋量参数 $\lambda_{\dot{\upsilon}}$ 是无约束的。显然 (131) 的规范变换构成一个群，即 Λ 规范群。可以证明，W 的 Λ 变换,

$$e^{W'} = e^{W}e^{-\bar{\Lambda}},\ \bar{\Lambda} = \bar{\lambda}^{n}\partial_{n} + \bar{\lambda}^{\upsilon}\partial_{\upsilon} + \bar{\lambda}_{\dot{\upsilon}}\bar{\partial}^{\dot{\upsilon}}, \tag{132}$$

can be supplemented by certain transformations of F and N , such that the supervector field E_{α} , Eq. (123a), does not change. In the infinitesimal case, since $\delta\hat{E}_{\mu} = -e^{W}\left[\bar{\Lambda}, \partial_{\mu}\right]e^{-\bar{W}}$, these transformations are:

可以补充上 F 和 N 的特定变换，使得式 (123a) 中的超矢量场 E_{α} 保持不变。在无穷小情况下，由于 $\delta\hat{E}_{\mu} = -e^{W}\left[\bar{\Lambda}, \partial_{\mu}\right]e^{-\bar{W}}$ ，这些变换为:

$$\delta F = -\frac{1}{2}F\partial_{\mu}\bar{\lambda}^{\mu},\ \delta N_{\alpha}{}^{\mu} = -N_{\alpha}{}^{\upsilon}e^{W}\partial_{(\upsilon}\bar{\lambda}^{\mu)}. \tag{133}$$

The Gravitational Superfield

引力超场

Let us analyse the transformation of W under the K and Λ gauge groups, $\mathrm{e}^{W'} = \mathrm{e}^{K}\mathrm{e}^{W}\mathrm{e}^{-\overline{\Lambda}}$. In the infinitesimal case, this reduces to

我们来分析 W 在 K 和 Λ 规范群下的变换，$\mathrm{e}^{W'} = \mathrm{e}^{K}\mathrm{e}^{W}\mathrm{e}^{-\overline{\Lambda}}$。在无穷小情形下，这简化为

$$\delta W = \delta W^M \partial_M = K - \overline{\Lambda} + O(W)$$

$$= \left(K^m - \overline{\lambda}^m\right)\partial_m + \left(K^\mu - \overline{\lambda}^\mu\right)\partial_\mu + \left(K_{\dot\mu} - \overline{\lambda}_{\dot\mu}\right)\overline{\partial}^{\dot\mu} + O(W). \tag{134}$$

Here the vector parameter K^m is real but otherwise unconstrained, and the spinor parameters K^μ and $\overline{\lambda}_{\dot\mu}$ are unconstrained. Therefore, it is possible to choose a gauge condition

此处矢量参数 K^m 是实的，但除此之外没有约束，旋量参数 K^μ 和 $\overline{\lambda}_{\dot\mu}$ 也没有约束。因此，我们可以选取规范条件

$$W = -\mathrm{i}H,\ H = H^m\partial_m = \bar{H}. \tag{135}$$

A different gauge fixing is possible [49]. First one may gauge fix W to have no spinor components, $W = W^n\partial_n$, and then impose the additional condition

也可以采用不同的规范固定方法 [49]。首先可以将 W 规范固定为不含旋量分量，$W = W^n\partial_n$，之后再施加附加条件

$$\exp\left(\bar{W}^n\partial_n\right)x^m = x^m + \mathrm{i}\mathcal{H}^m\left(x,\theta,\overline{\theta}\right),\ \overline{\mathcal{H}}^m = \mathcal{H}^m. \tag{136}$$

Given a covariantly chiral superfield (129), it holds that

对于协变 chiral 超场 (129)，有如下关系

$$\Phi\left(x,\theta,\overline{\theta}\right) = \mathrm{e}^{\bar{W}}\widehat{\Phi}(x,\theta) = \widehat{\Phi}(x + \mathrm{i}\mathcal{H},\theta). \tag{137}$$

The residual gauge freedom, which preserves the condition (136), is determined by considering the variation

满足条件 (136) 的剩余规范自由度可通过考虑变分得到

$$\mathrm{i}\delta\mathcal{H}^m = \delta\mathrm{e}^{\bar{W}}x^m = K\mathrm{e}^{\bar{W}}x^m - \mathrm{e}^{\bar{W}}\Lambda x^m$$

$$= K^m - \mathrm{e}^{\bar{W}}\lambda^m + \mathrm{i}K^N\partial_N\mathcal{H}^m. \tag{138}$$

Here the right-hand side should be purely imaginary; hence, K^m is expressed in terms of λ^m and $\overline{\lambda}^m$ as follows:

此处右手边必须是纯虚数；因此，K^m 可通过 λ^m 和 $\overline{\lambda}^m$ 表示如下:

$$K^m = \frac{1}{2} e^{\bar{W}} \lambda^m + \frac{1}{2} e^W \overline{\lambda}^m = \frac{1}{2} \lambda^m (x + i\mathcal{H}, \theta) + \frac{1}{2} \overline{\lambda}^m \left(x - i\mathcal{H}, \overline{\theta} \right), \tag{139}$$

and the variation $\delta\mathcal{H}^m$ turns into

且变分 $\delta\mathcal{H}^m$ 变为

$$\delta\mathcal{H}^m = K^N \partial_N \mathcal{H}^m + \frac{i}{2} \left(\lambda^m (x + i\mathcal{H}, \theta) - \overline{\lambda}^m \left(x - i\mathcal{H}, \overline{\theta} \right) \right). \tag{140}$$

It is also necessary to require $\delta e^{\bar{W}} \theta^\mu = 0$ and $\delta e^{\bar{W}} \overline{\theta}_{\dot{\mu}} = 0$, which gives

我们还需要满足 $\delta e^{\bar{W}} \theta^\mu = 0$ 和 $\delta e^{\bar{W}} \overline{\theta}_{\dot{\mu}} = 0$， 由此得到

$$K^\mu = \lambda^\mu (x + i\mathcal{H}, \theta), \ \bar{K}_{\dot{\mu}} = \overline{\lambda}_{\dot{\mu}} \left(x - i\mathcal{H}, \overline{\theta} \right), \tag{141}$$

as well as $\Lambda^M = \left(\lambda^m (x, \theta), \lambda^\mu (x, \theta), e^{-\bar{W}} e^W \overline{\lambda}_{\dot{\mu}} \left(x, \overline{\theta} \right) \right)$. Substituting the obtained expressions for K^μ, K^μ and $\bar{K}_{\dot{\mu}}$ into (138), we arrive at the gauge transformation law of the gravitational superfield, Eq. (30).

以及 $\Lambda^M = \left(\lambda^m (x, \theta), \lambda^\mu (x, \theta), e^{-\bar{W}} e^W \overline{\lambda}_{\dot{\mu}} \left(x, \overline{\theta} \right) \right)$。将得到的 K^μ, K^μ 和 $\bar{K}_{\dot{\mu}}$ 表达式代入 (138)，我们就得到引力超场的规范变换定律，即式 (30)。

Chiral Prepotential

手征预势

In order to uncover a remaining prepotential, we first analyse the structure of R, F, and E. These objects are invariant under the local Lorentz transformations, and therefore we can compute them by imposing the gauge condition (126). In this gauge $E_\alpha = F\widehat{E}_\alpha, C_{\alpha\beta}{}^\gamma = 2E_{(\alpha} \ln F \delta_{\beta)}{}^\gamma$, and therefore the spinor connections are

为了找出剩余的预势，我们首先分析 R、F 和 E 的结构。这些对象在局域洛伦兹变换下不变，因此我们可以通过施加规范条件 (126) 计算它们。在该规范下，旋量联络因 $E_\alpha = F\widehat{E}_\alpha, C_{\alpha\beta}{}^\gamma = 2E_{(\alpha} \ln F \delta_{\beta)}{}^\gamma$ 为

$$\Omega_{\alpha\beta\gamma} = -2\varepsilon_{\alpha(\beta} E_{\gamma)} \ln F, \ \overline{\Omega}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} = -2\varepsilon_{\dot{\alpha}(\dot{\beta}} \bar{E}_{\dot{\gamma})} \ln \bar{F}. \tag{142}$$

Now we can evaluate the relation (97a) to end up with explicit expressions for the chiral torsion R and its conjugate $\bar{R}$:

现在我们可以计算关系式 (97a)，最终得到手征挠率 R 及其共轭 $\bar{R}$ 的显式表达式:

$$\bar{R} = -\frac{1}{4}\hat{E}^{\mu}\hat{E}_{\mu}F^2,\ R = -\frac{1}{4}\hat{\bar{E}}_{\dot{\mu}}\hat{\bar{E}}^{\dot{\mu}}\bar{F}^2. \tag{143}$$

Given a scalar superfield U , a short calculation gives

给定标量超场 U ，经过简单计算可得

$$\left(\bar{\mathcal{D}}^2 - 4R\right)U = \hat{\bar{E}}_{\dot{\mu}}\hat{\bar{E}}^{\dot{\mu}}\left(\bar{F}^2 U\right). \tag{144}$$

In order to compute $E^{-1} = \mathrm{Ber}\left(E_A{}^M\right)$, we introduce a semi-covariant frame

为了计算 $E^{-1} = \mathrm{Ber}\left(E_A{}^M\right)$ ，我们引入半协变标架

$$\hat{E}_A = \left(\hat{E}_a, \hat{E}_\alpha, \hat{\bar{E}}^{\dot{\alpha}}\right) = \hat{E}_A{}^M\partial_m,\ \hat{E}_a = -\frac{\mathrm{i}}{4}(\tilde{\sigma}_a)^{\dot{\alpha}\alpha}\left\{\hat{E}_\alpha, \hat{\bar{E}}_{\dot{\alpha}}\right\}, \tag{145}$$

which is constructed in terms of the prepotential W^M and its conjugate, in accordance with (123). In the gauge (126), the inverse supervielbein E_A is related to $\hat{E}_A$ as follows:

它由预势 W^M 及其共轭按照式 (123) 构造得到。在规范 (126) 下，逆超标架 E_A 与 $\hat{E}_A$ 满足如下关系:

$$E_\alpha = F\hat{E}_\alpha,\ \bar{E}^{\dot{\alpha}} = \bar{F}\hat{\bar{E}}^{\dot{\alpha}}, \tag{146a}$$

$$\begin{aligned} E_a = F\bar{F}\hat{E}_a &+ \frac{\mathrm{i}}{4}F(\tilde{\sigma}_a)^{\dot{\alpha}\alpha}\left(\Omega_{\dot{\alpha}\alpha}{}^{\beta} - \delta_\alpha{}^\beta\bar{E}_{\dot{\alpha}}\ln F\right)\hat{E}_\beta \\ &+ \frac{\mathrm{i}}{4}\bar{F}(\tilde{\sigma}_a)^{\dot{\alpha}\alpha}\left(\overline{\Omega}_{\alpha\dot{\alpha}}{}^{\dot{\beta}} - \delta_{\dot{\alpha}}{}^{\dot{\beta}}E_\alpha\ln\bar{F}\right)\hat{\bar{E}}_{\dot{\beta}}. \end{aligned} \tag{146b}$$

It follows that

由此可得

$$E^{-1} = F^2\bar{F}^2\hat{E}^{-1},\ \hat{E}^{-1} = \mathrm{Ber}\left(\hat{E}_A{}^M\right). \tag{147}$$

So far, F appears to be unconstrained. However, it follows from the algebra of covariant derivatives that

到目前为止，F 似乎是无约束的。然而，根据协变导数的代数可以推出

$$(-1)^{\varepsilon_B}\mathcal{T}_{\alpha B}{}^B = 0, \tag{148}$$

while the direct evaluation of this structure gives (see, e.g., [35] for the technical details)

而对该结构直接计算可得 (技术细节参见例如文献 [35])

$$-(-1)^{\varepsilon_B}\mathcal{T}_{\alpha B}{}^{B} = E_\alpha \ln\left[E^{-1}F^2\left(1\cdot \mathrm{e}^{\overleftarrow{W}}\right)\right] = E_\alpha \ln\left[\widehat{E}^{-1}\bar{F}^2F^4\left(1\cdot \mathrm{e}^{\overleftarrow{W}}\right)\right], \tag{149}$$

where the operator $\overleftarrow{W}$ is defined by

其中算符 $\overleftarrow{W}$ 定义为

$$U\overleftarrow{W} = (-1)^{\varepsilon_M}\partial_M\left(UW^M\right) \Rightarrow \left(U\cdot \mathrm{e}^{\overleftarrow{W}}\right) = \left(1\cdot \mathrm{e}^{\overleftarrow{W}}\right)\mathrm{e}^{W}U. \tag{150}$$

We conclude that $\overline{\varphi}^{-3} := \bar{F}^2F^4\widehat{E}^{-1}\left(1\cdot \mathrm{e}^{W}\right)$ is annihilated by E_α . This result can be equivalently written as

我们得出结论: $\overline{\varphi}^{-3} := \bar{F}^2F^4\widehat{E}^{-1}\left(1\cdot \mathrm{e}^{W}\right)$ 被 E_α 零化。该结果也可以等价写为

$$\varphi^{-3} = F^2\bar{F}^4\widehat{E}^{-1}\left(1\cdot \mathrm{e}^{\overleftarrow{W}}\right),\ \bar{E}_{\dot\alpha}\varphi = 0. \tag{151}$$

By construction, the chiral superfield φ is nowhere vanishing. It follows that

根据构造，手征超场 φ 处处非零，因此可得

$$F = \varphi^{1/2}\overline{\varphi}^{-1}\left(1\cdot \mathrm{e}^{\overleftarrow{W}}\right)^{-1/3}\left(1\cdot \mathrm{e}^{\overleftarrow{\bar{W}}}\right)^{1/6}\widehat{E}^{1/6}, \tag{152}$$

and then Eq. (147) gives

随后式 (147) 给出

$$E = \overline{\varphi}\varphi\left[\widehat{E}\left(1\cdot \mathrm{e}^{\bar{W}}\right)\left(1\cdot \mathrm{e}^{\overleftarrow{\bar{W}}}\right)\right]^{1/3}. \tag{153}$$

The covariantly chiral superfield φ is called the chiral prepotential. It turns out that, modulo purely gauge degrees of freedom, the covariant derivatives are expressed in terms of W^M, φ , and their conjugates. These are the prepotentials for the GWZ superspace geometry. The transformation law of φ follows from (151)

协变手征超场 φ 被称为手征预势。可以证明，模去纯规范自由度后，协变导数可以通过 W^M, φ 及其共轭表示。这些就是 GWZ 超空间几何的预势。φ 的变换规律由式 (151) 得到

$$\delta\varphi^3 = K^M\partial_M\varphi^3 + \varphi^3\mathrm{e}^{\bar{W}}\left(\partial_m\lambda^m - \partial_\mu\lambda^\mu\right). \tag{154}$$

If we represent the chiral prepotential in the form

如果我们将手征预势写为如下形式

$$\varphi = \mathrm{e}^{\bar{W}}\widehat{\varphi},\ \bar{\partial}_{\dot\mu}\widehat{\varphi} = 0, \tag{155}$$

then the transformation law (154) will turn into

那么变换规律 (154) 就变为

$$\delta \widehat{\varphi}^3 = \lambda^M \partial_M \widehat{\varphi}^3 + \widehat{\varphi}^3 \left(\partial_m \lambda^m - \partial_\mu \lambda^\mu\right) = \partial_m \left(\lambda^m \widehat{\varphi}^3\right) - \partial_\mu \left(\lambda^\mu \widehat{\varphi}^3\right). \tag{156}$$

This is the transformation law of a chiral density.

这就是手征密度的变换定律。

By construction, the prepotential W^M is invariant under the super-Weyl transformations (111). It is a short calculation to see that the chiral prepotential φ transforms as

根据构造，预势 W^M 在超外尔变换 (111) 下不变。简单计算即可得知手征预势 φ 的变换形式为

$$\delta_\sigma \varphi = -\sigma \varphi \tag{157}$$

In conformal supergravity, the super-Weyl transformations belong to the gauge group. Making use of the super-Weyl gauge freedom allows one to impose the gauge $\varphi = 1$. Therefore, the gravitational superfield is the only prepotential in conformal supergravity, modulo purely gauge degrees of freedom.

在共形超引力中，超外尔变换属于规范群。利用超外尔规范自由度可以指定规范 $\varphi = 1$ 。因此，共形超引力中，除纯规范自由度外，引力超场是唯一的预势。

Chiral Action

手征作用量

It follows from the above analysis that E can be written as $E = \varphi^3 \bar{F}^2 \left(1 \cdot e^{\bar{W}}\right)$. In conjunction with the identity (144), the chiral action (78) can be rewritten as follows:

由上述分析可得，E 可写为 $E = \varphi^3 \bar{F}^2 \left(1 \cdot e^{\bar{W}}\right)$。结合恒等式 (144)，手征作用量 (78) 可重写为:

$$S_c = -4 \int d^4x \, d^2\theta d^2\bar{\theta} \varphi^3 \left(1 \cdot e^{\bar{W}}\right) \mathcal{L}_c \frac{\bar{F}^2 \gamma}{\widehat{\bar{E}}_{\dot{\mu}} \widehat{\bar{E}}^{\dot{\mu}} \left(\bar{F}^2 \gamma\right)}, \tag{158}$$

We now recall the well-known result for a change of variable in superspace [49] (see [35] for a pedagogical derivation). Given a first-order differential operator $K = K^N \partial_N$, it holds that

现在我们回顾超空间中变量替换的著名结果 [49], 教学式推导见 [35]。给定一阶微分算子 $K = K^N \partial_N$ ，有如下结论

$$z'^M = e^K z^M \Rightarrow \mathrm{Ber}\left(\partial_M z'^N\right) = \left(1 \cdot e^K\right), \tag{159}$$

and therefore $\int dz' L(z') = \int dz \left(1 \cdot e^K\right) e^K L(z)$. As follows from (129), the covariantly chiral superfields depend on chiral variables $\tilde{x}^m$ and $\tilde{\theta}^\mu$,

因此 $\int \mathrm{d}z' L(z') = \int \mathrm{d}z \left(1 \cdot \mathrm{e}^{K}\right) \mathrm{e}^{K} L(z)$。由 (129) 可知，协变手征超场依赖于手征变量 $\tilde{x}^m$ 和 $\tilde{\theta}^\mu$，

$$\hat{\bar{E}}_{\dot{\mu}} \Phi = 0 \Rightarrow \ \Phi(z) = \hat{\Phi}(\tilde{x}, \tilde{\theta}), \ \tilde{z}^M = \left(\tilde{x}^m, \tilde{\theta}^\mu, \tilde{\bar{\theta}}_{\dot{\mu}}\right) = \mathrm{e}^{\bar{W}} z^M. \tag{160}$$

In the variables $\tilde{z}^M$, the operator $\hat{\bar{E}}^{\dot{\mu}}$ becomes a partial derivative, $\hat{\bar{E}}^{\dot{\mu}} = \partial / \partial \tilde{\bar{\theta}}_{\dot{\mu}}$. Now, making use of (159) in (158) leads to the following simple result:

在变量 $\tilde{z}^M$ 下，算符 $\hat{\bar{E}}^{\dot{\mu}}$ 变为偏导数 $\hat{\bar{E}}^{\dot{\mu}} = \partial / \partial \tilde{\bar{\theta}}_{\dot{\mu}}$。现在将 (159) 代入 (158) 可得到如下简洁结果:

$$S_{\mathrm{c}} = \int \mathrm{d}^4 x \ \mathrm{d}^2 \theta \hat{\varphi}^3 \hat{\mathcal{L}}_{\mathrm{c}} \tag{161}$$

Here we have denoted $\tilde{x}^m$ and $\tilde{\theta}^\mu$ simply as x^m and θ^μ. This result shows that the chiral integration measure in (76) is

这里我们将 $\tilde{x}^m$ 和 $\tilde{\theta}^\mu$ 简记为 x^m 和 θ^μ。该结果表明，(76) 中的手征积分测度为

$$\mathcal{E} = \varphi^3, \tag{162}$$

and this interpretation agrees with the transformation law (156).

且该解释与变换规律 (156) 一致。

Matter Multiplets in Conformal Supergravity

共形超引力中的物质多重态

In this section, we introduce the most popular matter multiplets and describe several famous models for them. Practically, all results will be presented in conformal superspace. They can be recast in terms of the $U(1)$ or GWZ superspace geometries by making use of the degauging formalism described in section ”Other Superspace Formulations for Conformal Supergravity.”

本节我们将介绍最常用的物质多重态，并阐述几个相关的著名模型。实际上，所有结果都将在共形超空间中给出，利用“共形超引力的其他超空间表述”一节中介绍的退规范形式，可以将它们改写为 $U(1)$ 或 GWZ 超空间几何的形式。

Scalar Multiplet

标量多重态

The minimal scalar multiplet was introduced by Wess and Zumino in their first paper on supersymmetry [50]. In conformal superspace, minimal scalar multiplets are described in terms of covariantly chiral primary scalar superfields. Such a superfield ϕ obeys the constraints $K^B\phi = 0$ and $\overline{\nabla}^{\dot{\alpha}}\phi = 0$. In general, every covariantly chiral primary superfield ϕ of definite dimension Δ satisfies equation (66). If we do not assume ϕ to be an eigenvector of $\mathbb{D}$, then it must hold that

极小标量多重态由韦斯和朱米诺在他们第一篇关于超对称的论文 [50] 中提出。在共形超空间中，极小标量多重态用协变 chiral 原初标量超场描述。这类超场 ϕ 满足约束条件 $K^B\phi = 0$ 和 $\overline{\nabla}^{\dot{\alpha}}\phi = 0$ 。一般而言，每个具有确定量纲 Δ 的协变 chiral 原初超场 ϕ 都满足方程 (66)。如果我们不假设 ϕ 是 $\mathbb{D}$ 的本征向量，那么必然有

$$K^B\phi = 0,\ \overline{\nabla}^{\dot{\beta}}\phi = 0 \ \Rightarrow\ \mathbb{Y}\phi = -\frac{2}{3}\mathbb{D}\phi. \tag{163}$$

The superfield ϕ contains three independent component fields, which can be chosen as follows: $\varphi := \phi|, \eta_\alpha := \nabla_\alpha\phi|$ and $F := -\frac{1}{4}\nabla^2\phi\,|$. In theories with at most two derivatives at the component level, the complex scalar F is an auxiliary field.

超场 ϕ 包含三个独立分量场，可选取为: $\varphi := \phi|, \eta_\alpha := \nabla_\alpha\phi|$ 和 $F := -\frac{1}{4}\nabla^2\phi\,|$ 。在分量层面最多包含二阶导数的理论中，复标量 F 是一个辅助场。

As a simple example of a supergravity-matter system, we consider a curved superspace extension of the massless Wess-Zumino model [50]. It corresponds to choosing a canonical dimension for the chiral scalar, $\mathbb{D}\phi = \phi$. The action is

作为超引力-物质系统的简单例子，我们考虑无质量韦斯-朱米诺模型 [50] 的弯曲超空间推广。它对应于为 chiral 标量选取正则量纲，即 $\mathbb{D}\phi = \phi$ 。其作用量为

$$S_{\rm WZ} = \int \mathrm{d}^{4|4}z E\bar{\phi}\phi + \left\{\frac{\lambda}{3}\int \mathrm{d}^4x\ \mathrm{d}^2\theta\mathcal{E}\phi^3 +\ \text{c.c.}\right\}, \tag{164}$$

with λ a complex coupling constant.

其中 λ 是一个复耦合常数。

Superconformal Sigma Model

超共形西格玛模型

In Minkowski superspace $\mathbb{M}^{4|4}$, general $\mathcal{N} = 1$ supersymmetric two-derivative theories of scalar multiplets are nonlinear σ -models, which are described by chiral scalar superfields ϕ^I and their conjugates $\bar{\phi}^{\bar{I}}$ taking their values in an arbitrary Kähler manifold $\mathcal{M}$ [51]. In supergravity, however, σ -model couplings turn out to be more restrictive, and the target space $\mathcal{M}$ must be a Kähler-Hodge manifold [52]. Within the locally superconformal setting, this means that we have to consider a superconformal sigma model on a Kähler cone (see, e.g., [28, 53] for a more detailed discussion). Here, our goal is to show how these restrictions emerge.

在闵氏超空间 $\mathbb{M}^{4|4}$ 中，一般的标量多重态 $\mathcal{N}=1$ 超对称两导数理论是非线性 σ 模型，由手征标量超场 ϕ^I 及其共轭 $\overline{\phi}^{\bar{I}}$ 描述，二者取值于任意凯勒流形 $\mathcal{M}$ [51]。然而在超引力中，σ 模型耦合的约束更强，目标空间 $\mathcal{M}$ 必须为凯勒-霍奇流形 [52]。在局域超共形框架下，这意味着我们需要研究凯勒锥上的超共形 σ 模型 (更详细的讨论参见例如文献 [28, 53])。本文的目标是说明这些约束是如何产生的。

Let $N\left(\phi,\overline{\phi}\right)$ be the Kähler potential of $\mathcal{M}$ and $g_{I\bar{J}}=\partial_I\partial_{\bar{J}}N\equiv N_{I\bar{J}}$ its Kähler metric. We start with a naive curved-superspace extension of the $\mathcal{N}=1$ supersymmetric σ -model action (An overall minus sign is inserted in (165) in order to give the correct sign for the Einstein-Hilbert term at the component level, if (165) is viewed as the supergravity-matter action (see [28] for more details).)

设 $N\left(\phi,\overline{\phi}\right)$ 是 $\mathcal{M}$ 的凯勒势，$g_{I\bar{J}}=\partial_I\partial_{\bar{J}}N\equiv N_{I\bar{J}}$ 是它的凯勒度量。我们从 $\mathcal{N}=1$ 超对称 σ 模型作用量的朴素弯曲超空间推广开始 (若将 (165) 视为超引力-物质作用量，为了在分量层面给出爱因斯坦-希尔伯特项的正确符号，我们在 (165) 中插入了一个整体负号，更多细节参见文献 [28]。)

$$S=-\int \mathrm{d}^{4|4}zEN\left(\phi,\overline{\phi}\right),\ K^B\phi^I=0,\ \overline{\nabla}^{\dot{\beta}}\phi^I=0. \tag{165}$$

Here the dynamical variables ϕ^I are postulated to be covariantly chiral primary scalar superfields. Since ϕ^I are local holomorphic coordinates, $\mathbb{D}\phi^I$ and $\mathbb{Y}\phi^I$ must be holomorphic vector fields on $\mathcal{M}$,

此处动力学变量 ϕ^I 被假定为协变手征主标量超场。由于 ϕ^I 是局部全纯坐标，$\mathbb{D}\phi^I$ 与 $\mathbb{Y}\phi^I$ 必须是 $\mathcal{M}$ 上的全纯向量场，

$$\mathbb{D}\phi^I=\chi^I(\phi)\Leftrightarrow\mathbb{Y}\phi^I=-\frac{2}{3}\chi^I(\phi), \tag{166}$$

where we have used (163). The action (165) must be locally superconformal. Then, in accordance with (74), N must be neutral under the $\mathrm{U}(1)_R$ group and have dimension +2, and therefore

此处我们使用了式 (163)。作用量 (165) 必须是局域超共形的。因此根据式 (74)，N 在 $\mathrm{U}(1)_R$ 群下必须是中性的，且量纲为 +2，由此可得

$$\chi^I(\phi)\partial_I N\left(\phi,\overline{\phi}\right)=\overline{\chi}^{\bar{I}}\left(\overline{\phi}\right)\overline{\partial}_{\bar{I}}N\left(\phi,\overline{\phi}\right), \tag{167a}$$

$$\chi^I(\phi)\partial_I N\left(\phi,\overline{\phi}\right)=\overline{\chi}^{\bar{I}}\left(\overline{\phi}\right)\overline{\partial}_{\bar{I}}N\left(\phi,\overline{\phi}\right)=N\left(\phi,\overline{\phi}\right). \tag{167b}$$

Differentiating the condition $\chi^I\partial_I N=N$ with respect to $\overline{\partial}_{\bar{J}}$ gives

对条件 $\chi^I\partial_I N=N$ 关于 $\overline{\partial}_{\bar{J}}$ 求导可得

$$\chi^I(\phi)g_{I\bar{J}}\left(\phi,\overline{\phi}\right)=\overline{\partial}_{\bar{J}}N\left(\phi,\overline{\phi}\right)=\overline{\chi}_{\bar{J}}\left(\phi,\overline{\phi}\right)\Rightarrow\chi^I(\phi)=g^{I\bar{J}}\overline{\partial}_{\bar{J}}N. \tag{168}$$

The obtained relations have several nontrivial implications. First of all, the equations (167b) and (168) imply that N is a globally defined function on $\mathcal{M}$,

得到的关系有若干非平凡推论。首先，方程 (167b) 和 (168) 表明 N 是 $\mathcal{M}$ 上整体定义的函数，

$$N = g_{I\bar{J}}\chi^I\overline{\chi}^{\bar{J}}. \tag{169}$$

Therefore, the Kähler two-form, $\Omega = 2\mathrm{i}g_{I\bar{J}}\mathrm{d}\phi^I \wedge \mathrm{d}\overline{\phi}^{\bar{J}}$, is exact; hence, $\mathcal{M}$ is necessarily non-compact. Secondly, it follows that χ^I is a homothetic conformal Killing vector field

因此，凯勒二形式 $\Omega = 2\mathrm{i}g_{I\bar{J}}\mathrm{d}\phi^I \wedge \mathrm{d}\overline{\phi}^{\bar{J}}$ 是恰当的；故 $\mathcal{M}$ 必定是非紧致的。其次，由此可得 χ^I 是位似共形克利福德向量场

$$\nabla_I\chi^J = \delta_I^J,\ \overline{\nabla}_{\bar{I}}\chi^J = \bar{\partial}_{\bar{I}}\chi^J = 0. \tag{170}$$

The sigma model (165) can be generalised to include a superpotential

sigma 模型 (165) 可以推广为包含超势的形式

$$S = -\int \mathrm{d}^{4|4}zEN\left(\phi,\overline{\phi}\right) + \left\{\int \mathrm{d}^4x\ \mathrm{d}^2\theta\mathcal{E}W(\phi) + \text{ c.c.}\right\}. \tag{171}$$

Here $W(\phi)$ is a holomorphic scalar field on the target space. It should obey $\mathbb{D}W(\phi) = 3W(\phi)$ and $\mathbb{Y}W(\phi) = -2W(\phi)$, which imply the homogeneity condition

此处 $W(\phi)$ 是目标空间上的全纯标量场。它需满足 $\mathbb{D}W(\phi) = 3W(\phi)$ 和 $\mathbb{Y}W(\phi) = -2W(\phi)$ ，这二者导出了齐次条件

$$\chi^I(\phi)\,\partial_I W(\phi) = 3W(\phi). \tag{172}$$

It should be mentioned that local complex coordinates in $\mathcal{M}$ can be chosen in such a way that $\chi^I(\phi) = \phi^I$. Then, the Kähler potential $N\left(\phi,\overline{\phi}\right)$ obeys the following homogeneity condition:

需要指出的是，可以在 $\mathcal{M}$ 中选取局部复坐标使得 $\chi^I(\phi) = \phi^I$ 成立。此时凯勒势 $N\left(\phi,\overline{\phi}\right)$ 满足如下齐次性条件:

$$\phi^I\partial_I N\left(\phi,\overline{\phi}\right) = N\left(\phi,\overline{\phi}\right). \tag{173}$$

Superconformal Higher-Derivative Sigma Model

超共形高阶导数西格玛模型

Here we discuss a higher-derivative superconformal σ -model, which was originally introduced in the GWZ superspace as an induced action [54]. It appears that its uplift to conformal superspace cannot be given solely in terms of the conformally covariant derivatives ∇_A and should explicitly involve connection superfields.

我们在此讨论一个高阶导数超共形 σ 模型，它最初是作为诱导作用量在 GWZ 超空间中引入的 [54]。该模型提升到共形超空间后无法仅用共形协变导数 ∇_A 表示，必须显式包含联络超场。

Let $K\left(\phi^I, \overline{\phi}^{\bar{J}}\right)$ be the Kähler potential of an arbitrary Kähler manifold $\mathcal{M}$. We introduce a higher-derivative locally supersymmetric theory described in terms of covariantly chiral scalar superfields $\phi^I, \overline{\mathcal{D}}^{\dot{\alpha}} \phi^I = 0$, which are neutral under the super-Weyl transformations, $\delta_\sigma \phi^I = 0$. The higher-derivative action proposed in [54] is

设 $K\left(\phi^I, \overline{\phi}^{\bar{J}}\right)$ 为任意凯勒流形 $\mathcal{M}$ 的凯勒势。我们引入一个由协变手征标量超场 $\phi^I, \overline{\mathcal{D}}^{\dot{\alpha}} \phi^I = 0$ 描述的高阶导数局域超对称理论，这些超场在超外尔变换下是中性的，即 $\delta_\sigma \phi^I = 0$。文献 [54] 提出的高阶导数作用量为

$$S = \frac{1}{16} \int \mathrm{d}^{4|4} z E \Big\{ g_{I\bar{J}}\left(\phi, \overline{\phi}\right) \Big[\mathfrak{D}^2 \phi^I \overline{\mathfrak{D}}^2 \overline{\phi}^{\bar{J}} - 8 G_{\alpha\dot{\alpha}} \mathcal{D}^\alpha \phi^I \overline{\mathcal{D}}^{\dot{\alpha}} \overline{\phi}^{\bar{J}} \Big]$$

$$+ F_{IJ\bar{K}\bar{L}}\left(\Phi, \overline{\Phi}\right) \mathcal{D}^\alpha \Phi^I \mathcal{D}_\alpha \Phi^J \overline{\mathcal{D}}_{\dot{\alpha}} \overline{\Phi}^{\bar{K}} \overline{\mathcal{D}}^{\dot{\alpha}} \overline{\Phi}^{\bar{L}} \Big\}, \tag{174}$$

where $g_{I\bar{J}} = \partial_I \partial_{\bar{J}} K$ is the Kähler metric, $\mathfrak{D}^2 \phi^I$ is defined as follows (The operator $\mathfrak{D}^2$ in (175) should not be confused with $\mathfrak{D}^\alpha \mathfrak{D}_\alpha$ in U (1) superspace.)

其中 $g_{I\bar{J}} = \partial_I \partial_{\bar{J}} K$ 是凯勒度量，$\mathfrak{D}^2 \phi^I$ 定义如下 (请注意不要将 (175) 中的算符 $\mathfrak{D}^2$ 与 U (1) 超空间中的 $\mathfrak{D}^\alpha \mathfrak{D}_\alpha$ 混淆)

$$\mathfrak{D}^2 \phi^I := \mathcal{D}^2 \phi^I + \Gamma^I_{KL} \mathcal{D}^\alpha \phi^K \mathcal{D}_\alpha \phi^L, \tag{175}$$

and $F_{IJ\bar{K}\bar{L}}$ is a tensor field on the target space that is constructed from the Kähler metric $g_{I\bar{J}}$, the Riemann tensor $R_{I\bar{J}K\bar{L}}$, and, in general, its covariant derivatives. A typical expression for $F_{IJ\bar{K}\bar{L}}$ is

且 $F_{IJ\bar{K}\bar{L}}$ 是目标空间上由凯勒度量 $g_{I\bar{J}}$ 、黎曼张量 $R_{I\bar{J}K\bar{L}}$ 以及 (一般情况下) 其协变导数构造出的张量场。$F_{IJ\bar{K}\bar{L}}$ 的典型表达式为

$$F_{IJ\bar{K}\bar{L}} = \alpha_1 R_{(I\bar{K}J)\bar{L}} + \alpha_2 g_{(I\bar{K}g_J)\bar{L}} + \dots, \tag{176}$$

with α_1 and α_2 numerical coefficients. We recall that the Christoffel symbols Γ^I_{KL} and the curvature $R_{I\bar{J}K\bar{L}}$ are given by the expressions

其中 α_1 和 α_2 为数值系数。我们回顾一下，克里斯托费尔符号 Γ^I_{KL} 和曲率 $R_{I\bar{J}K\bar{L}}$ 由下式给出

$$\Gamma^I_{JK} = g^{I\bar{L}} \partial_J \partial_K \partial_{\bar{L}} K, R_{I\bar{J}K\bar{L}} = \partial_I \partial_K \partial_{\bar{J}} \partial_{\bar{L}} K - g^{M\bar{N}} \partial_I \partial_K \partial_{\bar{N}} K \partial_{\bar{J}} \partial_{\bar{L}} \partial_M K \tag{177}$$

It is an instructive exercise to show that the action (174) is super-Weyl invariant. This action is manifestly invariant under Kähler transformations

证明 (174) 的作用量具有超外尔不变性是一个有益的练习。该作用量在凯勒变换下具有明显的不变性

$$K\left(\phi,\overline{\phi}\right) \rightarrow K\left(\phi,\overline{\phi}\right)+\Lambda(\phi)+\overline{\Lambda}\left(\overline{\phi}\right), \tag{178}$$

with $\Lambda(\phi)$ being an arbitrary holomorphic function.

其中 $\Lambda(\phi)$ 是任意全纯函数。

The super-Weyl invariance of (174) may be traced to the existence of a super-conformal operator Δ introduced in [55]. In conformal superspace, this operator is defined to act on a primary chiral weight-zero scalar $\overline{\phi}$ as

(174) 的超外尔不变性可追溯到文献 [55] 中引入的超共形算符 Δ 。在共形超空间中，该算符被定义为作用在权为零的初态手征标量 $\overline{\phi}$ 上，满足

$$\Delta\overline{\phi}=-\frac{1}{64}\overline{\nabla}^{2}\nabla^{2}\overline{\nabla}^{2}\overline{\phi} \tag{179a}$$

and the resulting weight-three chiral superfield is primary,

得到的权为三的手征超场是初态场,

$$K^{B}\overline{\phi}=0,\ \nabla^{\beta}\overline{\phi}=0,\ \mathbb{D}\overline{\phi}=0\ \Rightarrow\ K^{B}\Delta\overline{\phi}=0,\ \overline{\nabla}^{\dot{\beta}}\Delta\overline{\phi}=0.$$

(179b)

Degauging $\Delta\overline{\phi}$ to the GWZ superspace gives

将 $\Delta\overline{\phi}$ 退规范到 GWZ 超空间后得到

$$\Delta\overline{\phi}\ :=\ -\frac{1}{64}\left(\overline{\mathcal{D}}^{2}-4R\right)\left\{\mathcal{D}^{2}\overline{\mathcal{D}}^{2}\overline{\phi}+8\mathcal{D}^{\alpha}\left(G_{\alpha\dot{\alpha}}\overline{\mathcal{D}}^{\dot{\alpha}}\overline{\phi}\right)\right\}. \tag{180}$$

The super-Weyl transformation law of this superfield is $\delta_{\sigma}\Delta\overline{\phi}=3\sigma\Delta\overline{\phi}$. For any covariantly chiral scalars ϕ and ψ , it holds that

该超场的超外尔变换规律为 $\delta_{\sigma}\Delta\overline{\phi}=3\sigma\Delta\overline{\phi}$ 。对任意协变手征标量 ϕ 和 ψ ，都有

$$\int \mathrm{d}^{4}x\mathrm{d}^{2}\theta\mathcal{E}\psi\Delta\overline{\phi}=\frac{1}{16}\int \mathrm{d}^{4|4}zE\left\{\mathcal{D}^{2}\psi\bar{\mathcal{D}}^{2}\overline{\phi}-8\mathcal{D}^{\alpha}\psi G_{\alpha\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\overline{\phi}\right\}. \tag{181}$$

If the chiral scalars ϕ and ψ are inert under the super-Weyl transformations, this functional is super-Weyl invariant.

如果手征标量 ϕ 和 ψ 在超外尔变换下不发生变化，该泛函就是超外尔不变的。

The operator (179) is a supersymmetric generalisation of the conformal fourth-order scalar operator in curved space

算符 (179) 是弯曲空间中共形四阶标量算符的超对称推广

$$\Delta_0 = \Box\Box - \nabla^a \left(2\mathfrak{R}_{ab}\nabla^b - \frac{2}{3}\mathfrak{R}\nabla_a\right), \ \Box = \nabla^a\nabla_a \tag{182}$$

discovered by Fradkin and Tseytlin [56]. Here ∇_a denotes the torsion-free Lorentz covariant derivative, with $\mathcal{R}_{ab}$ and $\mathcal{R}$ being its Ricci tensor and scalar curvature, respectively. The operator (179) was constructed for the first time in [55] using the conformal superspace approach, although there had been earlier attempts to construct such an operator (see the discussion in [57]). Making use of its degauged form, Eq. (180), a new representation for the nonlocal action generating the super-Weyl anomalies was derived in [57].

由弗拉德金和谢伊特林发现 [56]。此处 ∇_a 表示无挠洛伦兹协变导数，$\mathcal{R}_{ab}$ 和 $\mathcal{R}$ 分别是其里奇张量和标量曲率。尽管此前已有研究者尝试构造这类算符 (相关讨论见文献 [57])，算符 (179) 最早仍是在文献 [55] 中利用共形超空间方法构造得到的。借助其退规范形式即式 (180)，文献 [57] 推导出了生成超外尔反常的非局域作用量的新表示。

Tensor Multiplet

张量多重态

The massless tensor multiplet was introduced by Siegel [58] as a dual version of the minimal scalar multiplet. In conformal superspace, it is described by a primary covariantly chiral spinor superfield η_α of dimension 3/2 ,

无质量张量多重态由 Siegel[58] 引入，是极小标量多重态的对偶版本。在共形超空间中，它由维数为 3/2 的基本协变手征旋量超场 η_α 描述,

$$K^B\eta_\alpha = 0, \ \bar{\nabla}^{\dot\beta}\eta_\alpha = 0 \ \mathbb{D}\eta_\alpha = \frac{3}{2}\eta_\alpha, \tag{183}$$

which is defined modulo gauge transformations

其定义模去规范变换

$$\delta\eta_\alpha = -\frac{\mathrm{i}}{4}\bar{\nabla}^2\nabla_\alpha U, \ \bar{U} = U, \tag{184}$$

with the gauge parameter being a primary dimensionless real scalar. The descendant

其中规范参数是一个基本无量纲实标量。后代场

$$\mathbb{G} = \frac{1}{2}\left(\nabla^\alpha\eta_\alpha + \bar{\nabla}_{\dot\alpha}\bar{\eta}^{\dot\alpha}\right) = \bar{\mathbb{G}} \tag{185}$$

is a gauge-invariant field strength. It has the following properties:

是规范不变场强。它具有以下性质:

$$K^B\mathbb{G}=0,\ \overline{\nabla}^2\mathbb{G}=0,\ \mathbb{D}\mathbb{G}=2\mathbb{G}. \tag{186}$$

These constraints define a real linear multiplet. Such superfields were originally introduced by Ferrara, Wess, and Zumino [59] to describe flavour current multiplets.

这些约束定义了一个实线性多重态。这类超场最初由 Ferrara、Wess 和 Zumino[59] 引入，用于描述味流多重态。

The superconformal tensor multiplet is described by the action [60]

超共形张量多重态由如下作用量描述 [60]

$$S=-\int \mathrm{d}^{4|4}zE\mathbb{G}\ln\frac{\mathbb{G}}{\bar{\phi}\phi},\ K^B\phi=0,\ \overline{\nabla}^{\dot\beta}\phi=0,\ \mathbb{D}\phi=\phi. \tag{187}$$

Both $\mathbb{G}$ and ϕ are assumed to be nowhere vanishing. Dependence of the action (187) on ϕ and $\overline{\phi}$ is fictitious, since the action remains unchanged under transformations $\phi\to \mathrm{e}^{\sigma}\phi$, where σ is an arbitrary covariantly chiral weight-zero scalar. In the literature, (187) is referred to as the model for an improved tensor multiplet [60]. It is a unique superconformal representative in the family of tensor multiplet models introduced in [58].

假设 $\mathbb{G}$ 和 ϕ 处处非零。作用量 (187) 对 ϕ 和 $\overline{\phi}$ 的依赖是虚的，因为作用量在变换 $\phi\to \mathrm{e}^{\sigma}\phi$ 下保持不变，其中 σ 是任意协变手征零权标量。文献中将 (187) 称为改进张量多重态模型 [60]，它是文献 [58] 引入的张量多重态模型族中唯一的超共形代表。

Three-Form Multiplet

三形式多重态

Let us consider the representation (68) for $n=0$. The unconstrained prepotential ψ in (67) is necessarily complex for $w\neq 3$. In the $w=3$ case, however, one can impose the reality condition $\overline{\psi}=\psi=P$. This leads to the three-form multiplet (In global supersymmetry, the three-form multiplet was originally proposed by Gates [61].) described by the primary covariantly chiral scalar

我们来考虑 $n=0$ 的表示式 (68)。式 (67) 中的无约束预势 ψ 对于 $w\neq 3$ 必然是复的。然而，在 $w=3$ 的情形下，我们可以施加实条件 $\overline{\psi}=\psi=P$ 。这便得到了由基本协变手征标量描述的三形多重态 (整体超对称中的三形多重态最初由 Gates 提出 [61]。)

$$\Pi=-\frac{1}{4}\overline{\nabla}^2P,\ \bar{P}=P,\ K^BP=0,\ \mathbb{D}P=2P. \tag{188}$$

The main difference of the three-form multiplet from the minimal scalar multiplet is that the imaginary part of the auxiliary field $F:=-\frac{1}{4}\nabla^2\Pi\,|$ of Π is the field strength of a gauge three-form.

三形多重态与极小标量多重态的主要区别在于，Π 的辅助场 $F := -\frac{1}{4}\nabla^2\Pi\,|$ 的虚部是规范三形的场强。

The prepotential P in (188) is defined modulo gauge transformations $\delta P = \mathbb{G}$, where the gauge parameter is a real linear superfield, Eq. (185). The simplest super-conformal and gauge-invariant action to describe the dynamics of this multiplet is given by

(188) 式中的预势 P 是模规范变换 $\delta P = \mathbb{G}$ 定义的，其中规范参数为实线性超场，见式 (185)。描述该多重态动力学的最简单超共形规范不变作用量为

$$
\begin{aligned}
S &= \int \mathrm{d}^{4|4}zE\left\{\left(\overline{\Pi}\Pi\right)^{1/3} + 2\kappa P\right\} \\
&= \int \mathrm{d}^{4|4}zE\left(\overline{\Pi}\Pi\right)^{1/3} + \left\{\kappa\int \mathrm{d}^4x\, \mathrm{d}^2\theta\mathcal{E}\Pi + \text{ c.c. }\right\},
\end{aligned} \tag{189}
$$

where κ is a real coupling constant.

其中 κ 为实耦合常数。

Non-minimal Scalar Multiplet

非最小标量多重态

We next turn to a non-minimal scalar multiplet (In global supersymmetry, it was introduced by Gates and Siegel [62].). In conformal superspace it is described by a primary complex scalar superfield Γ satisfying the constraint

我们接下来讨论非最小标量多重态 (整体超对称中，它由 Gates 和 Siegel 引入 [62])。在共形超空间中，它由满足约束的原初复标量超场 Γ 描述

$$
K^B\Gamma = 0,\ \overline{\nabla}^2\Gamma = 0 \ \Rightarrow\ \mathbb{Y}\Gamma = \frac{2}{3}(2-\mathbb{D})\Gamma. \tag{190a}
$$

We choose to parametrise the dimension of Γ as

我们选择将 Γ 的维度参数化为

$$
\mathbb{D}\Gamma = \frac{2}{3n+1}\Gamma \ \Rightarrow\ \mathbb{Y}\Gamma = \frac{4n}{3n+1}\Gamma, \tag{190b}
$$

following the notation introduced in [6]. For $n \neq 0, -1/3$, the constraint (190) defines a complex linear superfield. In the $n = 0$ case, the $\mathrm{U}(1)_R$ charge of Γ is equal to zero, and Γ can be subject to the reality condition $\overline{\Gamma} = \Gamma$, which corresponds to the real linear multiplet (186). The general solution to the constraint (190) is

遵循文献 [6] 引入的记号。对 $n \neq 0, -1/3$ 而言，约束 (190) 定义了一个复线性超场。在 $n = 0$ 情形下，Γ 的 $\mathrm{U}(1)_R$ 荷等于零，且 Γ 可以满足实性条件 $\bar{\Gamma} = \Gamma$，对应实线性多重态 (186)。约束 (190) 的通解为

$$\Gamma = \bar{\nabla}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}, \tag{191}$$

where the unconstrained prepotential $\bar{\Psi}^{\dot{\alpha}}$ may be chosen to be primary. It is defined modulo gauge transformations $\delta\bar{\Psi}^{\dot{\alpha}} = \bar{\nabla}_{\dot{\beta}} \bar{\lambda}^{(\dot{\alpha}\dot{\beta})}$, where the gauge parameter may be chosen to be primary.

其中无约束预势 $\bar{\Psi}^{\dot{\alpha}}$ 可以取为原初的。它在规范变换 $\delta\bar{\Psi}^{\dot{\alpha}} = \bar{\nabla}_{\dot{\beta}} \bar{\lambda}^{(\dot{\alpha}\dot{\beta})}$ 下是模定义的，规范参数也可以取为原初的。

A unique superconformal model, which is constructed solely in term of Γ and $\bar{\Gamma}$ and involves at most two derivatives at the component level, is given by

唯一仅由 Γ 和 $\bar{\Gamma}$ 构造、分量层面最多包含两阶导数的超共形模型由下式给出

$$S = -\frac{1}{n} \int \mathrm{d}^{4|4} z E \left(\bar{\Gamma}\Gamma\right)^{(3n+1)/2}. \tag{192}$$

In global supersymmetry, it was observed by Deo and Gates [63] that the complex linear constraint $\bar{D}^2\Gamma = 0$ admits a deformation $\bar{D}^2\Gamma = -4\Xi$ in the presence of a chiral scalar Ξ . This idea is compatible with local superconformal symmetry. Indeed, in conformal superspace the constraint (190) can be deformed to define the following improved linear constraint

在整体超对称中，Deo 和 Gates[63] 发现，当存在手征标量 Ξ 时，复线性约束 $\bar{D}^2\Gamma = 0$ 允许形变 $\bar{D}^2\Gamma = -4\Xi$ 。这个想法与局域超共形对称性相容。事实上，在共形超空间中，约束 (190) 可以形变，得到如下改进线性约束

$$-\frac{1}{4}\bar{\nabla}^2 \Upsilon = \Xi, \ K^B \Xi = 0, \ \bar{\nabla}^{\dot{\beta}} \Xi = 0, \ \mathbb{D}\Xi = \frac{3\,(n+1)}{3n+1}\Xi. \tag{193}$$

In general, Ξ may be a function of matter chiral scalars, $\Xi = \Xi\,(\phi)$ (see [63, 64]). Such constraints naturally arise in the framework of the $\mathcal{N} = 1$ superfield description of $\mathcal{N} = 2$ supersymmetric sigma models [65].

一般而言，Ξ 可以是物质手征标量 $\Xi = \Xi\,(\phi)$ 的函数 (见 [63, 64])。这类约束自然出现在 $\mathcal{N} = 2$ 超对称 sigma 模型的 $\mathcal{N} = 1$ 超场描述框架中 [65]。

It follows from (193) that the choice $n = -1$ is special in the sense that Ξ becomes dimensionless, and therefore, one can impose the superconformal constraint

由 (193) 可知，选择 $n = -1$ 是特殊的，此时 Ξ 变为无量纲，因此我们可以施加超共形约束

$$K^B \Upsilon = 0, \ -\frac{1}{4}\bar{\nabla}^2 \Upsilon = \mu = \text{ const } \Rightarrow \mathbb{D}\Upsilon = -\Upsilon. \tag{194}$$

This multiplet originates as the compensator of the non-minimal AdS supergravity proposed in [46]. It is also used to describe the dynamics of a Goldstino [66].

该多重态最初是文献 [46] 提出的非最小反德西特超引力的补偿场，也被用来描述戈德斯提诺的动力学 [66]。

Vector Multiplet

向量多重态

The Abelian vector multiplet was introduced by Wess and Zumino in their first paper on supersymmetry [50]. Its Yang-Mills extension was derived by Ferrara and Zumino [67] and, independently, by Salam and Strathdee [68]. Here we briefly review the conformal superspace formulation for the Abelian vector multiplet and related superconformal models.

阿贝尔向量多重态由韦斯和朱米诺在他们关于超对称的第一篇论文中引入 [50]。其杨-米尔斯推广由费拉拉和朱米诺 [67] 独立完成，萨拉姆和斯特拉思迪也独立得到了该结果 [68]。在此我们简要回顾阿贝尔向量多重态的共形超空间表述以及相关的超共形模型。

The Abelian vector multiplet is described by a scalar dimension-zero prepotential V defined modulo gauge transformations of the form

阿贝尔向量多重态由标量零维预势 V 描述，其定义差以下形式的规范变换

$$\delta_\Lambda V = \Lambda + \overline{\Lambda},\ \overline{\nabla}_{\dot\alpha}\Lambda = 0. \tag{195}$$

Both the prepotential V and the chiral gauge parameter may be chosen to be primary. Associated with V is the primary chiral spinor descendant

预势 V 和手征规范参数都可以取为初态场。与 V 关联的是初态手征旋子 descendant 场

$$W_\alpha = -\frac{1}{4}\overline{\nabla}^2\nabla_\alpha V,\ K^B W_\alpha = 0,\ \overline{\nabla}^{\dot\beta} W_\alpha = 0,\ \mathbb{D}W_\alpha = \frac{3}{2}W_\alpha, \tag{196}$$

which is gauge invariant, $\delta_\Lambda W_\alpha = 0$. The field strength W_α is a reduced chiral superfield in the sense that it obeys the reality condition $\nabla^\alpha W_\alpha = \overline{\nabla}_{\dot\alpha}\bar{W}^{\dot\alpha} \equiv \nabla W$. It should be pointed out that ∇W is a primary dimension-2 superfield. Modulo purely gauge degrees of freedom, the independent components of V can be chosen as follows: $\eta_\alpha = W_\alpha\Big|, v_{\alpha\dot\alpha} = \frac{1}{2}\left[\nabla_\alpha, \overline{\nabla}_{\dot\alpha}\right]V\Big|$, and $D = -\frac{1}{2}\nabla W\,|$.

它是规范不变的，$\delta_\Lambda W_\alpha = 0$ 。场强 W_α 是约化手征超场，满足实条件 $\nabla^\alpha W_\alpha = \overline{\nabla}_{\dot\alpha}\bar{W}^{\dot\alpha} \equiv \nabla W$ 。需要指出，∇W 是二维初态超场。扣除纯规范自由度后，V 的独立分量可如下选取：$\eta_\alpha = W_\alpha\Big|, v_{\alpha\dot\alpha} = \frac{1}{2}\left[\nabla_\alpha, \overline{\nabla}_{\dot\alpha}\right]V\Big|$，以及 $D = -\frac{1}{2}\nabla W\,|$。

Dynamics of the free vector multiplet is described by the action [8]:

自由向量多重态的动力学由如下作用量描述 [8]:

$$S = \frac{1}{4}\int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathcal{E} W^2 + \text{ c.c.} \tag{197}$$

For a single vector multiplet, this is a unique locally superconformal action with at most two derivatives at the component level. In the case that $(\nabla W)^{-1}$ exists, nonlinear superconformal actions exist of the form [69]

对于单个向量多重态，这是分量层面最多含两阶导数的唯一局域超共形作用量。当 $(\nabla W)^{-1}$ 存在时，存在如下形式的非线性超共形作用量 [69]

$$S = \frac{1}{4}\int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathcal{E} W^2 + \text{ c.c. } + \frac{1}{4}\int \mathrm{d}^{4|4}z E \frac{W^2\bar{W}^2}{(\nabla W)^2}\mathfrak{H}(u,\bar{u}), \tag{198}$$

where $u := \frac{1}{8}\nabla^2\left[W^2(\nabla W)^{-2}\right]$ is a primary dimensionless antichiral superfield and $\mathfrak{H}(z,\bar{z})$ is a real function of a complex variable. This family includes a unique $\mathrm{U}(1)$ duality-invariant theory [70, 71]

其中 $u := \frac{1}{8}\nabla^2\left[W^2(\nabla W)^{-2}\right]$ 是无量纲初态反手征超场，$\mathfrak{H}(z,\bar{z})$ 是复变量的实函数。该族包含唯一的 $\mathrm{U}(1)$ 对偶不变理论 [70, 71]

$$S = \frac{1}{4}\cosh\gamma \int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathcal{E} W^2 + \text{ c.c. } + \frac{1}{4}\sinh\gamma \int \mathrm{d}^{4|4}z E \frac{W^2\bar{W}^2}{(\nabla W)^2\sqrt{u\bar{u}}}, \tag{199}$$

where the coupling constant γ must be nonnegative [70] (The general formalism for U(1) duality-invariant $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetric theories was developed in [72].). This nonlinear extension of the supersymmetric Maxwell action (197) is called the super ModMax theory.

其中耦合常数 γ 必须非负 [70](U(1) 对偶不变 $\mathcal{N}=1$ 和 $\mathcal{N}=2$ 超对称理论的一般形式体系已在文献 [72] 中建立)。这种超对称麦克斯韦作用量 (197) 的非线性推广被称为超 ModMax 理论。

Within the GWZ superspace formalism, the action (199) can be rewritten in a simpler, albeit not manifestly superconformal form, originally given in [70,71]

在 GWZ 超空间形式体系中，作用量 (199) 可以改写为更简单的形式 (尽管不是明显超共形的)，该形式最早由文献 [70,71] 给出

$$S = \frac{1}{4}\cosh\gamma \int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathcal{E} W^2 + \text{ c.c. } + \frac{1}{4}\sinh\gamma \int \mathrm{d}^{4|4}z E \frac{W^2\bar{W}^2}{\sqrt{\mathbf{u}\bar{\mathbf{u}}}}, \tag{200}$$

where $\mathbf{u} := \frac{1}{8}\mathcal{D}^2 W^2$. In order to make direct contact with the $\mathrm{U}(1)$ duality-invariant formalism of [72], this action can be rewritten in the form

其中 $\mathbf{u} := \frac{1}{8}\mathcal{D}^2 W^2$ 。为了直接对应文献 [72] 的 $\mathrm{U}(1)$ 对偶不变形式体系，该作用量可改写为如下形式

$$S = \frac{1}{4}\int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathcal{E} W^2 + \text{ c.c. } + \frac{1}{4}\int \mathrm{d}^{4|4}z E W^2 \bar{W}^2 \Lambda\left(\mathbf{u}, \bar{\mathbf{u}}\right), \tag{201a}$$

$$\Lambda\left(\mathbf{u}, \bar{\mathbf{u}}\right) = \frac{\sinh\gamma}{\sqrt{\mathbf{u}\bar{\mathbf{u}}}} + \frac{1}{2}(1-\cosh\gamma)\left(\frac{1}{\mathbf{u}} + \frac{1}{\bar{\mathbf{u}}}\right). \tag{201b}$$

A large class of other interesting, and not necessarily superconformal, models for supersymmetric non-linear electrodynamics are based on deforming the super-Maxwell action (197) by a self-interaction $\int \mathrm{d}^{4|4}zE\mathcal{L}$, where

> 一大类其他有趣的 (不一定是超共形的) 超对称非线性电动力学模型，都是基于对超麦克斯韦作用量 (197) 添加自相互作用项 $\int \mathrm{d}^{4|4}zE\mathcal{L}$ 得到的，其中

$$\mathcal{L} = W^2\bar{W}^2\mathcal{H}\left(\omega, \bar{\omega}, \nabla W, \mathfrak{C}\right), \ \omega := \frac{1}{8}\nabla^2\left[W^2\mathfrak{C}^{-2}\right], \tag{202a}$$

$$K^B\mathfrak{C} = 0, \ \mathbb{D}\mathfrak{C} = 2\mathfrak{C}, \ \bar{\mathfrak{C}} = \mathfrak{C}. \tag{202b}$$

Here $\mathfrak{C}$ is a conformal compensator associated to an off-shell Poincaré supergravity (see the next section), while the composite $\mathcal{H}$ is constrained to be a real primary superfield of dimension -4. Well-known theories of this type are, for instance, the supersymmetric Born-Infeld theory [73], and the generalised Fayet-Iliopoulos terms in supergravity without gauged R -symmetry, which were recently introduced in [74].

> 此处 $\mathfrak{C}$ 是与离壳庞加莱超引力关联的共形补偿器 (见下一节)，而复合场 $\mathcal{H}$ 被约束为-维实初态超场。这类著名理论包括例如超对称博恩-因费尔德理论 [73]，以及近期文献 [74] 中引入的、未规范化 R 对称性的超引力推广法耶-伊波利奥项。

The supersymmetric Yang-Mills multiplet is well reviewed in the literature, see, e.g., [13, 17, 28, 35] , and its description in conformal superspace does not bring in new features. We refer the interested reader to the literature (see [75]).

> 超对称杨-米尔斯多重态在文献中已有充分综述，例如参见 [13, 17, 28, 35]，其在共形超空间中的描述并未带来新性质。感兴趣的读者可参阅相关文献 (参见 [75])。

Off-Shell Models for Pure Supergravity

纯超引力的离壳模型

As discussed in the introduction, there are several off-shell formulations for pure supergravity, including the old minimal [7-10], new minimal [11, 12], and non-minimal [4-6] theories. Here we present their formulations in conformal superspace. Due to space limitations, a discussion of general supergravity-matter systems is beyond the scope of this review.

> 正如引言中所述，纯超引力存在多种离壳表述，包括旧最小化理论 [7-10]、新最小化理论 [11, 12] 和非最小化理论 [4-6]。本文我们给出这些理论在共形超空间中的表述。受篇幅限制，对一般超引力-物质系统的讨论超出了本综述的范围。

As discussed in section "Conformal Superspace," conformal superspace can be identified with a pair $\left(\mathcal{M}^{4|4}, \nabla\right)$. In the case of Poincaré or AdS supergravity, the superspace geometric setup is a triple $\left(\mathcal{M}^{4|4}, \nabla, \mathfrak{C}\right)$, where $\mathfrak{C}$ is a compensator. The latter is a primary constrained scalar superfield such that (i) $\mathfrak{C}$ is nowhere vanishing (more precisely, $\mathfrak{C}^{-1}$ exists) and (ii) the dimension of $\mathfrak{C}$ is nonzero. These conditions imply that the local scale and local $\mathrm{U}(1)_R$ gauge freedom can be used to impose the gauge condition $\mathfrak{C} = 1$. If the compensator is real, $\overline{\mathfrak{C}} = \mathfrak{C}$, the required gauge condition is achieved by applying a local scale transformation.

正如“共形超空间”一节所讨论的，共形超空间可以等同于一对 $\left(\mathcal{M}^{4|4}, \nabla\right)$ 。对于庞加莱或反德西特超引力，超空间几何结构是三元组 $\left(\mathcal{M}^{4|4}, \nabla, \mathfrak{C}\right)$ ，其中 $\mathfrak{C}$ 是补偿场。后者是带约束的原初标量超场，满足:(i) $\mathfrak{C}$ 处处非零 (更准确地说，$\mathfrak{C}^{-1}$ 存在)；(ii) $\mathfrak{C}$ 的量纲非零。这些条件表明，我们可以利用定域标度和定域 $\mathrm{U}(1)_R$ 规范自由度施加规范条件 $\mathfrak{C} = 1$ 。若补偿场是实场 $\overline{\mathfrak{C}} = \mathfrak{C}$ ，则可通过一次定域标度变换得到所需的规范条件。

Old Minimal Supergravity

旧最小超引力

In the old minimal formulation for supergravity, the compensator is a nowhere vanishing primary chiral scalar ϕ , Eq. (66), of nonzero dimension Δ . Since ϕ^{-1} exists, the primary chiral scalar $\Phi = \phi^{1/\Delta}$ is also nowhere vanishing and its dimension is canonical, $\mathbb{D}\Phi = \Phi$. It is Φ and its conjugate $\overline{\Phi}$ which are chosen as the compensators in old minimal supergravity. Since the Weyl multiplet has 8 +8 off-shell degrees of freedom and the minimal scalar multiplet $\left(\Phi, \overline{\Phi}\right)$ contains $4+4$ fields, the multiplet of old minimal supergravity describes $12+12$ off-shell fields.

在超引力的旧最小形式中，补偿器是一个处处不为零的基本手征标量 ϕ (见式 (66))，其具有非零的维度 Δ 。由于 ϕ^{-1} 存在，基本手征标量 $\Phi = \phi^{1/\Delta}$ 也处处不为零，并且其维度是正则的，为 $\mathbb{D}\Phi = \Phi$ 。在旧最小超引力中，正是 Φ 及其共轭 $\overline{\Phi}$ 被选作补偿器。由于外尔多重态具有 $8+8$ 个脱壳自由度，并且最小标量多重态 $\left(\Phi, \overline{\Phi}\right)$ 包含 $4+4$ 个场，所以旧最小超引力的多重态描述了 $12+12$ 个脱壳场。

The action functional for pure old minimal supergravity is given by

纯旧最小超引力的作用量泛函由下式给出

$$S_{\text{old-minimal}} = -3 \int \mathrm{d}^{4|4} z E \overline{\Phi} \Phi + \left\{ \mu \int \mathrm{d}^4 x\, \mathrm{d}^2 \theta \mathcal{E} \Phi^3 + \text{ c.c.} \right\}, \tag{203}$$

where μ is a complex constant parameter. The choice $\mu = 0$ corresponds to Poincaré supergravity. For $\mu \neq 0$ the action describes AdS supergravity. Let us analyse the equations of motion for this theory. The chirality constraint on Φ and its equation of motion can be written as

其中 μ 是复常数参数。选择 $\mu = 0$ 对应庞加莱超引力，当 $\mu \neq 0$ 时该作用量描述反德西特超引力。我们来分析该理论的运动方程。Φ 上的手征约束及其运动方程可以写为

$$\overline{\nabla}_{\dot{\alpha}} \Phi^3 = 0, \tag{204a}$$

$$-\frac{1}{4}\bar{\nabla}^2\left(\bar{\Phi}\Phi^{-2}\right)=\mu. \tag{204b}$$

The equation of motion corresponding to the gravitational superfield proves to be

对应引力超场的运动方程为

$$\left[\nabla_\alpha,\bar{\nabla}_{\dot{\alpha}}\right]\left(\bar{\Phi}\Phi\right)^{-1/2}=0. \tag{204c}$$

In general, given a primary real scalar L of dimension -1, $\mathbb{D}L=-L$, its real vector descendant $\left[\nabla_\alpha,\bar{\nabla}_{\dot{\alpha}}\right]L$ is primary.

一般来说，给定量纲为 $-1, \mathbb{D}L=-L$ 的本原实标量 L，其本原实矢量后裔 $\left[\nabla_\alpha,\bar{\nabla}_{\dot{\alpha}}\right]L$ 是本原的。

The equations (204) can be degauged to U (1) superspace, which results in

方程 (204) 可以退规范到 U (1) 超空间，结果为

$$\bar{\mathcal{D}}_{\dot{\alpha}}\Phi^3=0, \tag{205a}$$

$$-\frac{1}{4}\left(\bar{\mathcal{D}}^2-4R\right)\left(\bar{\Phi}\Phi^{-2}\right)=\mu, \tag{205b}$$

$$\left\{G_{\alpha\dot{\alpha}}+\left[\mathcal{D}_\alpha,\bar{\mathcal{D}}_{\dot{\alpha}}\right]\right\}\left(\bar{\Phi}\Phi\right)^{-1/2}=0. \tag{205c}$$

Now, the super-Weyl and local $\mathrm{U}(1)_R$ gauge freedom can be used to impose the gauge condition $\Phi=1$. This implies that the $\mathrm{U}(1)_R$ connection vanishes and U (1) superspace geometry reduces to the GWZ geometry. The supergravity equations (205b) and (205c) turn into

现在，可以利用超外尔和局域 $\mathrm{U}(1)_R$ 规范自由度施加规范条件 $\Phi=1$。这意味着 $\mathrm{U}(1)_R$ 联络为零，且 U (1) 超空间几何约化为 GWZ 几何。超引力方程 (205b) 和 (205c) 变为

$$R=\mu,\ G_{\alpha\dot{\alpha}}=0, \tag{206}$$

and all information about the dynamics of supergravity is encoded in the super-Weyl tensor $W_{\alpha\beta\gamma}$. By analogy with the terminology used in general relativity, the equations (206) define an Einstein superspace.

超引力动力学的所有信息都编码在超外尔张量 $W_{\alpha\beta\gamma}$ 中。类比广义相对论中使用的术语，方程 (206) 定义了爱因斯坦超空间。

A unique maximally supersymmetric solution of (206) is characterised by the condition $W_{\alpha\beta\gamma}=0$. It is called $\mathcal{N}=1$AdS superspace and can be identified with the homogeneous space $\mathrm{OSp}(1\mid 4)/\mathrm{SO}(3,1)$ introduced in [76,77]. The superfield representations of AdS supersymmetry were classified by Ivanov and Sorin [78].

(206) 有唯一的极大超对称解，其特征为条件 $W_{\alpha\beta\gamma}=0$ 。它被称为 $\mathcal{N}=1$AdS 超空间，可以等同于文献 [76,77] 中引入的齐性空间 OSp (1 | 4) / SO (3, 1) 。反德西特超对称性的超场表示由伊万诺夫和索林 [78] 分类。

New Minimal Supergravity

新极小超引力

In the new minimal formulation for Poincaré supergravity, the compensator is a tensor multiplet $\mathbb{G}$ obeying the constraints (186). The supergravity action is given by the functional

在庞加莱超引力的新极小表述中，补偿场是满足约束条件 (186) 的张量多重态 $\mathbb{G}$ 。超引力作用量由如下泛函给出

$$S_{\text{new-minimal}}=3\int \mathrm{d}^{4|4}zE\mathbb{G}\ln\frac{\mathbb{G}}{\bar{\phi}\phi}, \tag{207}$$

which differs only by a negative overall factor from (187). The equation of motion for the chiral spinor prepotential η_α , Eq. (183), is

它仅比式 (187) 多一个整体负号差。手征旋量预备势 η_α 的运动方程即式 (183) 为

$$\bar{\nabla}^2\nabla_\alpha\ln\frac{\mathbb{G}}{\bar{\phi}\phi}=0, \tag{208}$$

and its general solution is given by

其通解为

$$\mathbb{G}=\bar{\Phi}\Phi,\ K^B\Phi=0,\ \bar{\nabla}^{\dot{\beta}}\Phi=0,\ \mathbb{D}\Phi=\Phi. \tag{209}$$

Here the chiral scalar Φ is nowhere vanishing. Now, the constraint $\bar{\nabla}^2\mathbb{G}=0$ leads to the equation on $\bar{\Phi}$

此处手征标量 Φ 处处非零。现在，约束条件 $\bar{\nabla}^2\mathbb{G}=0$ 给出关于 $\bar{\Phi}$ 的方程

$$\bar{\nabla}^2\bar{\Phi}=0, \tag{210}$$

which is equivalent to the equation (204b) with $\mu=0$. The equation of motion for the gravitational superfield can be shown to be equivalent to

它等价于含 $\mu=0$ 的式 (204b)。可以证明引力超场的运动方程等价于

$$\nabla_\alpha\ln\mathbb{G}\bar{\nabla}_{\dot{\alpha}}\ln\mathbb{G}-\left[\nabla_\alpha,\bar{\nabla}_{\dot{\alpha}}\right]\ln\mathbb{G}=0, \tag{211}$$

where the left-hand side proves to be a primary real vector superfield. This equation may be seen to be equivalent to (204c) if the representation (209) for $\mathbb{G}$ in terms of Φ is used.

其中左侧被证明是一个本原实矢量超场。如果使用 Φ 表示的 $\mathbb{G}$ 表达式 (209)，可以看出该方程等价于式 (204c)。

The above results imply that new minimal supergravity is classically equivalent to old minimal supergravity without cosmological term. The linear multiplet describes the same number of degrees of freedom, $4+4$, as the minimal scalar multiplet. Therefore, new minimal supergravity contains $12+12$ fields.

上述结果表明，新极小超引力在经典层面等价于无宇宙学项的旧极小超引力。线性多重态和极小标量多重态具有相同的自由度数目 $4+4$ ，因此新极小超引力包含 12+12 个场。

Three-Form Supergravity

三形式超引力

The only difference of three-form supergravity from the old minimal theory considered earlier is that the chiral compensator Φ is realised in the former theory as $\Phi=\Pi^{1/3}$, where Π is the chiral field strength of the three-form multiplet, Eq. (188). The supergravity action is

三形式超引力与前文讨论的旧极小理论的唯一区别是，手性补偿场 Φ 在三形式超引力中实现为 $\Phi=\Pi^{1/3}$ ，其中 Π 是三形式多重态的手性场强，见式 (188)。超引力作用量为

$$S_{\text{three-form}}=\int \mathrm{d}^{4|4}zE\Big\{-3\big(\bar{\Pi}\Pi\big)^{1/3}+2mP\Big\}, \tag{212}$$

with m a real coupling constant. The equation of motion for P takes the simplest form in terms of $\Phi=\Pi^{1/3}$ and its conjugate:

其中 m 是实耦合常数。P 的运动方程用 $\Phi=\Pi^{1/3}$ 及其复共轭可表示为最简形式:

$$-\frac{1}{4}\bar{\nabla}^2\left(\bar{\Phi}\Phi^{-2}\right)-\frac{1}{4}\nabla^2\left(\Phi\bar{\Phi}^{-2}\right)=2m. \tag{213}$$

This is equivalent to the equation

这等价于方程

$$-\frac{1}{4}\bar{\nabla}^2\left(\bar{\Phi}\Phi^{-2}\right)=\mu=\ \text{const},\quad \mathrm{Re}\,\mu=m \tag{214}$$

which has the same form as (204b). The new feature of the supergravity theory (212) is that the imaginary part of μ is now generated dynamically. It may be shown that the equation of motion for the gravitational superfield is equivalent to (205c). As a result, the supergravity theories (203) and (212) are classically equivalent.

其形式与 (204b) 相同。超引力理论 (212) 的新特点是 μ 的虚部现在由动力学产生。可以证明，引力超场的运动方程等价于 (205c)。因此，超引力理论 (203) 和 (212) 在经典层面等价。

The existence of three-form supergravity was first pointed out by Gates and Siegel [62]. Unlike the standard formulation of old minimal supergravity, the remarkable feature of three-form supergravity is that it allows a consistent coupling to the four-dimensional supermembrane [79] as demonstrated by Ovrut and Waldram [80] who built on the results of [43]. Within the GWZ formalism, the action (212) was presented in [81].

三形式超引力的存在最早由 Gates 和 Siegel 指出 [62]。与旧极小超引力的标准表述不同，三形式超引力的显著特点是，它可以与四维超膜实现一致耦合 [79]，这一点由 Ovrut 和 Waldram 在 [43] 的结果基础上证明 [80]。在 GWZ 形式体系下，作用量 (212) 最早出现在文献 [81] 中。

Non-minimal Supergravity

非极小超引力

In the non-minimal formulation for Poincaré supergravity, the compensator is a complex linear superfield Γ constrained by (190). The supergravity action is described by the functional

在庞加莱超引力的非极小表述中，补偿子是受式 (190) 约束的复线性超场 Γ 。超引力作用量由如下泛函描述

$$S_{\text{non-minimal}} = \frac{1}{n}\int \mathrm{d}^{4|4}z E\left(\bar{\Gamma}\Gamma\right)^{(3n+1)/2}, \tag{215}$$

which differs from (192) by an overall sign. It may be shown that non-minimal supergravity is classically equivalent to old minimal supergravity without cosmological term (see [13,35] for reviews). No supersymmetric cosmological term is allowed in non-minimal supergravity with the compensator Γ [13].

它与式 (192) 仅相差一个整体符号。可以证明，非极小超引力在经典层面等价于无宇宙学项的旧极小超引力，综述见文献 [13,35]。在采用补偿子 Γ 的非极小超引力中，不存在超对称宇宙学项 [13]。

To describe non-minimal AdS supergravity [46], the compensator Υ is chosen to obey the constraints (194), and the action is

为描述非极小反德西特 (AdS) 超引力 [46]，要求补偿子 Υ 满足约束条件 (194)，其作用量为

$$S_{\text{non-minimal AdS}} = -\int \mathrm{d}^{4|4}z E\left(\bar{\Upsilon}\Upsilon\right)^{-1}. \tag{216}$$

In order to derive the equation of motion for Υ , we note that $\delta\Upsilon$ is a complex linear superfield and hence $\delta\Upsilon = \bar{\nabla}_{\dot{\alpha}}\delta\bar{\Psi}^{\dot{\alpha}}$. Varying the action gives

为推导 Υ 的运动方程，我们注意到 $\delta\Upsilon$ 是复线性超场，因此有 $\delta\Upsilon = \bar{\nabla}_{\dot{\alpha}}\delta\bar{\Psi}^{\dot{\alpha}}$ 。对作用量变分得

$$\overline{\nabla}_{\dot\alpha}\left(\mathrm{Y}^2\overline{\mathrm{Y}}\right)^{-1} = 0 \;\Rightarrow\; \left(\mathrm{Y}^2\overline{\mathrm{Y}}\right)^{-1} = \Phi^3. \tag{217}$$

We see that the equation of motion for Y in the non-minimal theory (216) is equivalent to the off-shell constraint (204a) in old minimal supergravity. It follows that $\mathrm{Y} = \overline{\Phi}\Phi^{-2}$, and the off-shell constraint (194) turns into the equation of motion (204b) in old minimal supergravity. Finally, it may be shown that, in the non-minimal theory (216), the equation of motion for the gravitational superfield is equivalent to (204c), once Y is expressed in terms of Φ and its conjugate. We conclude that the minimal and non-minimal formulations for AdS supergravity, which are described by the actions (203) and (216), are classically equivalent.

我们可以看到，非极小理论 (216) 中 Y 的运动方程等价于旧极小超引力中的脱壳约束 (204a)。由此可得 $\mathrm{Y} = \overline{\Phi}\Phi^{-2}$ ，且脱壳约束 (194) 转化为旧极小超引力中的运动方程 (204b)。最后可以证明，在非极小理论 (216) 中，当 Y 通过 Φ 及其复共轭表示时，引力超场的运动方程等价于 (204c)。我们的结论是，由作用量 (203) 和 (216) 分别描述的 AdS 超引力极小表述与非极小表述在经典层面等价。

The complex linear multiplet describes $12 + 12$ degrees of freedom. Therefore, non-minimal supergravity contains $20 + 20$ off-shell fields.

复线性多重态描述 $12 + 12$ 个自由度。因此非极小超引力包含 $20 + 20$ 个脱壳场。

Conformal Supergravity

共形超引力

The conformal supergravity action [18, 82, 83] is

共形超引力作用量 [18, 82, 83] 为

$$S_{\mathrm{CSG}} = -\frac{1}{4}\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathcal{E}\, W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} + \text{ c.c.} \tag{218}$$

The corresponding equation of motion is

对应的运动方程是

$$\mathcal{B}_{\alpha\dot\alpha} = \overline{\mathcal{B}}_{\alpha\dot\alpha} = 0, \tag{219}$$

with $\mathcal{B}_{\alpha\dot\alpha}$ being the super-Bach tensor (70). This equation can be degauged to U (1) superspace to take the form

其中 $\mathcal{B}_{\alpha\dot\alpha}$ 是超巴赫张量 (70)。该方程退规范到 U (1) 超空间后可写为如下形式

$$\mathrm{i}\mathcal{D}_{\beta\dot\alpha}\mathcal{D}_\gamma W^{\alpha\beta\gamma} + \mathcal{D}_\beta\left(G_{\gamma\dot\alpha}W^{\alpha\beta\gamma}\right) = \mathrm{i}\mathcal{D}_{\alpha\dot\beta}\overline{\mathcal{D}}_{\dot\gamma}\bar W^{\dot\alpha\dot\beta\dot\gamma} - \overline{\mathcal{D}}_{\dot\beta}\left(G_{\alpha\dot\gamma}\bar W^{\dot\alpha\dot\beta\dot\gamma}\right) = 0.$$

(220)

It follows from the Bianchi identities (93c) and (98c) that every solution of the equations of motion for pure AdS supergravity (206) is also a solution of the equations of motion for conformal supergravity.

由比安基恒等式 (93c) 与 (98c) 可得，纯反德西特超引力 (206) 运动方程的每一个解，同时也都是共形超引力运动方程的解。

Acknowledgments We thank S. James Gates Jr. and Konstantinos Koutrolikos for their kind invitation to contribute this chapter to the Handbook of Quantum Gravity. We are grateful to Daniel Butter for useful discussions and to Stefan Theisen for comments on the manuscript. The work of SK is supported in part by the Australian Research Council, project No. DP200101944. The work of ER is supported by the Hackett Postgraduate Scholarship UWA, under the Australian Government Research Training Program. The work of GT-M is supported by the Australian Research Council (ARC) Future Fellowship FT180100353 and by the Capacity Building Package of the University of Queensland.

致谢我们感谢 S. James Gates Jr. 与 Konstantinos Koutrolikos 盛情邀请我们为《量子引力手册》撰写本章。我们感谢 Daniel Butter 提供的有益讨论，感谢 Stefan Theisen 对原稿提出的意见。SK 的工作部分得到澳大利亚研究委员会资助，项目编号 DP200101944。ER 的工作得到西澳大学哈克特研究生奖学金资助，该奖学金隶属于澳大利亚政府研究培训计划。GT-M 的工作得到澳大利亚研究委员会 (ARC) 未来研究员基金 FT180100353 与昆士兰大学能力建设项目资助。

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